DIFFERENTIAL GEOMETRY

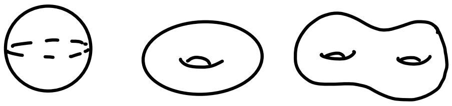
0 INTRODUCTION

Lecturer: Jack Smith (j.smith@dpmms.cam.ac.uk)

Two ways to think about manifolds:

1) Embedded manifolds: smoothly embedded subspaces in $\mathbb{R}^N$

e.g.



Extrinsic

or e.g. solns to equation: $\{x^2 = y^2+1\} \subset \mathbb{R}^2$

smooth ones

e.g. $SO(n) \subseteq \mathbb{R}^{n^2}$ ($M^TM = 1$, $\det M = 1$)

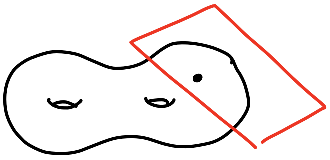
2) Abstract manifolds: (reasonable) topological space such that about each point p, $\exists$ local coordinates such that the coordinate transformations are smooth.

Intrinsic

will focus on abstract manifolds. But! Actually, the two definitions are equivalent.

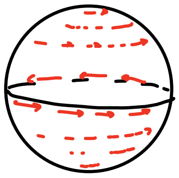
Basic constructions with manifolds

- Tangent space: linear approximation to manifold at some point: less obvious in abstract world.



- smooth maps between manifolds + derivatives

- Vector fields



and flow

- submanifolds (Embedded manifolds become submanifolds of $\mathbb{R}^N$).

- Could give manifold more structure and consider geometric consequences e.g. group structure (lie group).

  → tangent space at identity becomes a lie algebra

  → $\exists$ map from the lie algebra into the lie group itself (exp map).

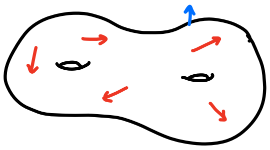
e.g. $GL(n, \mathbb{R})$, tangent space at id $=$ $\mathrm{Mat}_{n\times n}(\mathbb{R})$. Lie algebra structure: $[A,B] = AB - BA$.

exp map : $A \mapsto I + A + \frac{A^2}{2!} + \cdots$

Some questions we'll think about:

How do you differentiate a vector field?

- on $\mathbb{R}^n$ its easy (partial derivatives)
- what about on an embedded surface Σ in $\mathbb{R}^n$?

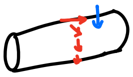


problem 1 · cant differentiate in directions out of surface.

problem 2 · if you differentiate along directions in surface, may end up with something pointing out of surface

NOT INTRINSIC

e.g



derivative points downwards

soln? extrinsic picture, go along surface + orthog project answer onto surface

seems reasonable, <u>BUT</u> this may depend then on the embedding

So this is a subtle question!

To answer this question, we'll use:

- tensors and differential forms
- Connections
- Parallel transport: moving a vector along a path s.t its derivative is zero.
- curvature.

A more abstract example

spacetime = manifold X

quantum particle described by a <u>wavefunction</u> $\psi : X \to \mathbb{C}$

what matters in $|\psi|$ and relative phases of ψ_1 and ψ_2.

Examples classes wednesdays

27 oct 1.30 - 3pm
10 NOV
24 NOV

1 MANIFOLDS AND SMOOTH MAPS

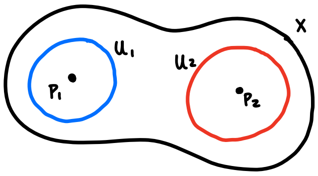
1.1 Manifolds

Dfn 1.1 a topological n-manifold is a topological space X s.t $\forall p \in X$, $\exists$ an open nhood $\mathcal{U}$ of p in X, an open set $V \subset \mathbb{R}^n$, and a homeomorphism $\varphi : \mathcal{U} \xrightarrow{\sim} V$

We also require X to be Hausdorff and second-countable

Hausdorff : for distinct points $p_1, p_2 \in X$, $\exists$ disjoint open U_1, U_2 s.t $p_1 \in U_1$, $p_2 \in U_2$

$U_1 \cap U_2 = \phi$



Second-countable: $\exists$ countable basis for the topology, i.e. $\exists$ countable collection of open sets U_i s.t. any open set is a union of the U_i.

Exm 1.2 : $\mathbb{R}^n$ is a topological manifold

- For any $p \in \mathbb{R}^n$, take $\mathcal{U} = \mathbb{R}^n$, and $\varphi = id : \mathbb{R}^n \to \mathbb{R}^n$.
- $\mathbb{R}^n$ is Hausdorff , e.g because its metrisable
- A countable basis is given by open balls with rational centre and rational radius.

Rem 1.3 (i) " Hausdorff + second-countable " is important but not restrictive in practice.

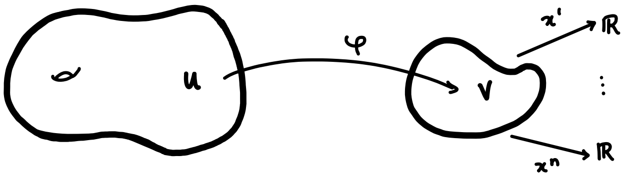
– For a space locally homeo to $\mathbb{R}^n$, it's equivalent to " X is metrisable and has countably many components"

(ii) The two conditions are inherited by subspaces.

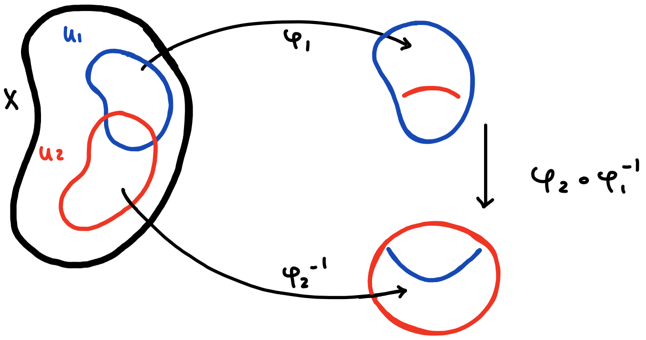
Exm 1.4: If X is a top n-manifold, then so is any open set $W \subset X$.

Given $p \in W$, pick $\varphi : \underset{X}{\underset{\cap}{\mathcal{U}}} \xrightarrow{\sim} \underset{\mathbb{R}^n}{\underset{\cap}{V}}$ from X. Then take $\varphi|_{U \cap W} : U \cap W \to \varphi(U \cap W)$ is a homeo.

Terminology:
- φ is called a **chart** about p.
- U is a **coordinate patch**
- If $x_1, \dots, x_n$ are the standard coordinates on $\mathbb{R}^n$, then $x_1 \circ \varphi, x_2 \circ \varphi, \dots, x_n \circ \varphi$ are called **local coordinates**



- The inverse of a chart is a **parametrization.**
- If $\varphi_1 : U_1 \xrightarrow{\sim} V_1$ and $\varphi_2 : U_2 \xrightarrow{\sim} V_2$ are two charts, the corresponding local coords $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are related by **transition function** $\varphi_2 \circ \varphi_1^{-1}$



Dfn 1.5 A map from an open subset of $\mathbb{R}^a$ to $\mathbb{R}^b$ is **smooth** if it has all partial derivatives of all orders

Given $f : X \to \mathbb{R}$

preliminary dfn: f is smooth if $f \circ \varphi^{-1}$ is smooth. for all charts φ, i.e. $f(x_1, \dots, x_n)$ is smooth as a function of local coordinates.

Dfn 1.6: An **atlas** for a topological n-manifold is a collection $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in \mathcal{A}}$ of charts that cover X ($\bigcup_{\alpha \in \mathcal{A}} U_\alpha = X$).

- An **atlas is smooth** if all transition functions $\varphi_\beta \cdot \varphi_\alpha^{-1}$ are smooth (as in Dfn 1.5).

- Given a smooth atlas $\mathbb{A}$, a **function $f : X \to \mathbb{R}$ is smooth** wrt $\mathbb{A}$ if $f \circ \varphi_\alpha^{-1}$ is smooth $\forall$ $\varphi_\alpha \in \mathbb{A}$.

Lem 1.7: f is smooth wrt $\mathbb{A}$ iff $\forall\, p \in X$, $\exists$ a chart φ_α about p such that $f \circ \varphi_\alpha^{-1}$ is smooth

pf: only if ✓

converse: take $\varphi_\beta : U_\beta \to V_\beta$. WTS $f \circ \varphi_\beta^{-1}$ is smooth. know $\forall\, p \in U_\beta$, $\exists\, \varphi_\alpha$ s.t $f \circ \varphi_\alpha^{-1}$ is smooth. But then near $\varphi_\beta(p)$, we have

$$f \circ \varphi_\beta^{-1} = \underbrace{\underset{\uparrow \atop \text{smooth}}{\left(f \circ \varphi_\alpha^{-1}\right)} \circ \underset{\uparrow \atop \text{smooth}}{\left(\varphi_\alpha \circ \varphi_\beta^{-1}\right)}}_{\mathbb{R}^n \text{ smoothness!}} \Rightarrow f \circ \varphi_\beta^{-1} \text{ is smooth.}$$

$\square$

Cor 1.8: Given a smooth atlas, all local coordinate functions are smooth.

Dfn 1.9: Two smooth atlases $\mathbb{A}$ and $\mathbb{B}$ are smoothly equivalent if $\mathbb{A} \cup \mathbb{B}$ is smooth.

- A smooth structure on X is an equivalence class of smooth atlases.
- A smooth n-manifold is a topological n-manifold equipped with a smooth structure.

Lem 1.10: If $\mathbb{A}$ and $\mathbb{B}$ are smooth atlases that are smoothly equivalent, then f is smooth wrt $\mathbb{A}$ iff its smooth wrt $\mathbb{B}$.

pf: Example sheet 1

Dfn 1.11: If X is a smooth manifold, then $f: X \to \mathbb{R}$ is smooth if its smooth wrt some (equivalently all) smooth atlases representing the smooth structure.

Exm 1.12
- $\mathbb{R}^n$ is a smooth n-manifold with smooth structure defined by the atlas $\{ \mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n \}$
- open subsets, as before
- If X, Y are smooth m-manifold, n-manifold then $X \times Y$ is a smooth $(m+n)$-manifold defined by product charts

Rem 1.13: (i) Being a topological n-manifold is a property of a topological space

(ii) Being a smooth manifold is a property plus a choice of smooth structure

(iii) For $n \leq 3$, every topological n-manifold admits a unique smooth structure

(iv) For $n \geq 4$, a topological n-manifold may admit no smooth structure (e.g. the E_8 4-manifold) or multiple different smooth structures (e.g. exotic S^7, exotic $\mathbb{R}^4$). But these results are hard!

Dfn 1.14: For a smooth n-manifold X, the integer n is the dimension of X : $\dim X$.

Note: you're free to add charts to your atlas, as long as they preserve smoothness.

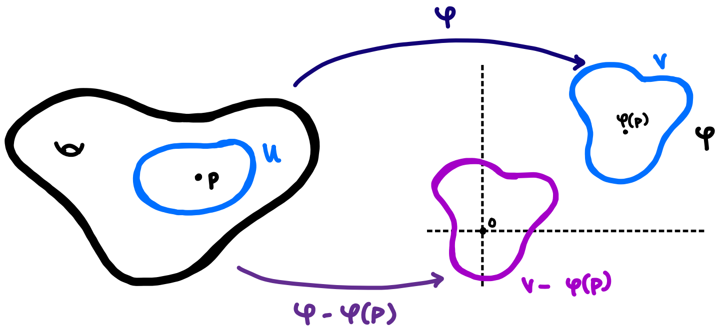
Example 1.15:

(i) given $p \in X$, and an open nhood W of p, we can always take / add a chart about p contained in W.

(ii) Can choose / add local coordinates about p such that p corresponds to the origin in these coords:

take any chart φ about p and consider $\varphi - \varphi(p)$.

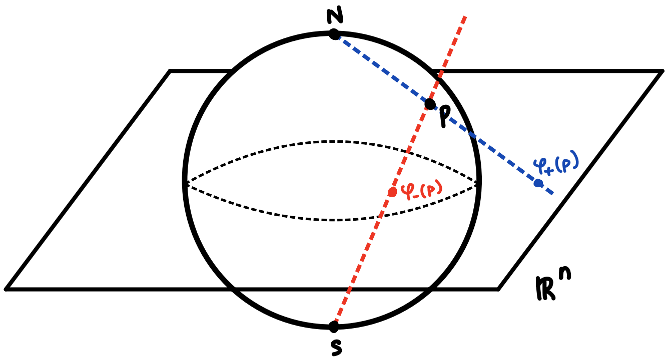
$[\ \varphi : \underset{\underset{X}{\cap}}{p \in U} \to \underset{\underset{\mathbb{R}^n}{\cap}}{V}$, $\varphi - \varphi(p)$ means subtract in $\mathbb{R}^n$



Exm 1.16: The n-sphere, S^n, is the n-manifold whose underlying top space is $\{ y \in \mathbb{R}^{n+1} : \|y\|^2 = 1 \} \subset \mathbb{R}^{n+1}$ with subspace topology. The smooth structure is defined by the following atlas:

There are two charts: $\varphi_\pm : U_\pm \xrightarrow{\sim} \mathbb{R}^n$, where $U_\pm = S^n \setminus \{(0,0,\dots,0,\pm1)\}$ (whole − north/south)

and $\varphi_\pm$ is stereographic projection $S^n \subset \mathbb{R}^{n+1}$



formula: $\varphi_\pm(y_1, \dots, y_{n+1}) = \frac{1}{1 \mp y_{n+1}} (y_1, \dots, y_n)$

Check: transition functions are smooth.

Local coordinates: $x^\pm$ satisfy $x_i^\pm = \frac{y_i}{1 \mp y_{n+1}}$

The height function y_{n+1} is smooth since its given by $y_{n+1} = \pm \frac{\|x^\pm\|^2 - 1}{\|x^\pm\|^2 + 1}$ on $U_\pm$.

1.2 Manifolds from sets

Observe: If X is a manifold, the charts know the topology in the sense that : a set $W \subset X$ is open iff $\varphi_\alpha(W)$ is open in $\mathbb{R}^n$ for all charts φ_α (φ_α are homeomorphisms)

(check)

Suppose we're given:

- a set X
- a collection $\{U_\alpha\}_{\alpha \in A}$ of sets covering X
- for each α, an open set $V_\alpha \subset \mathbb{R}^n$ and a bijection $\varphi_\alpha : U_\alpha \to V_\alpha$.

Suppose that $\forall \alpha, \beta$, the set $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in V_α (or $\mathbb{R}^n$), and the map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth ($\subset \mathbb{R}^n \to \subset \mathbb{R}^n$).

Dfn 1.17 (Non-standard) call such data a smooth pseudo-atlas on X, and the φ_α pseudo-charts.

Declare a set W in X to be open iff $\forall \alpha$, the set $\varphi_\alpha(U_\alpha \cap W)$ is open in $\mathbb{R}^n$.

Lemma 1.18 : This defines a topology on X

(check)

Prop 1.19: Apart from the possible failure of "Hausdorff and second countable", the resulting space is a topological n-manifold, and the pseudo-atlas is a smooth atlas (hence it defines a smooth structure).

pf: We need to check that each U_α is open and each φ_α is a homeomorphism , i.e. that $\forall\ W \subset U_\alpha$,

$$W \text{ open in } X \iff \varphi_\alpha(W) \text{ is open in } V_\alpha.$$

$\Rightarrow$: is obvious (check) we declared W to be open in X if $\forall\ \alpha$, $\varphi_\alpha(U_\alpha \cap W)$ is open in $\mathbb{R}^n$. But $W \subset U_\alpha$,
$\Rightarrow$ $\varphi_\alpha(U_\alpha \cap W) = \varphi_\alpha(W) \subset \varphi_\alpha(U_\alpha) = V_\alpha$ is open.

$\Leftarrow$: Suppose $\varphi_\alpha(W)$ is open . Then take any β , WTS $\varphi_\beta(W \cap U_\beta)$ is also open.

We have
$$\begin{aligned} \varphi_\beta(W \cap U_\beta) &= \varphi_\beta \circ \varphi_\alpha^{-1} \circ \varphi_\alpha(W \cap U_\beta) \\ &= (\varphi_\beta \circ \varphi_\alpha^{-1}) \circ \underbrace{\varphi_\alpha(W)}_{\text{open}} \cap \underbrace{\varphi_\alpha(U_\alpha \cap U_\beta)}_{\text{open}} \quad \text{using fact } \varphi_\alpha \text{ is bijection} \\ &= (\varphi_\alpha \circ \varphi_\beta^{-1})^{-1}(\text{open}) \\ &= \text{preimage of open set is open. under cont. map.} \end{aligned}$$

Say two smooth pseudo-atlases are equivalent if their union is a smooth pseudo-atlas.

Lemma 1.20 : Equivalent smooth pseudo atlases define the same manifold structure

Example 1.21 : The n-dimensional real projective space $\mathbb{RP}^n$ is the space of lines in $\mathbb{R}^{n+1}$.

- Any nonzero point in $\mathbb{R}^{n+1}$ defines a point $\langle x \rangle \in \mathbb{RP}^n$.
- All lines arise in this way
- $\langle x \rangle = \langle y \rangle \iff x = \alpha y$ for some $\alpha \in \mathbb{R} \setminus \{0\}$.

So we can label points of $\mathbb{RP}^n$ by the ratio $[x_0 : \cdots : x_n]$ called homogeneous coordinates.

Define the following pseudo charts :
For $i = 0, \dots, n$, let

$$U_i = \{[x_0 : \cdots : x_n] : x_i \neq 0\}$$

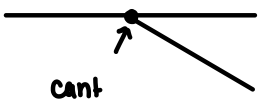
and define a bijection $\varphi_i : U_i \to \mathbb{R}^n$ by $[x_0 : \cdots : x_n] \mapsto \dfrac{(x_1, \dots, \hat{x}_i, \dots, x_n)}{x_i}$

This is a smooth pseudo-atlas and makes $\mathbb{RP}^n$ into a smooth manifold (example sheet 1).

note: change $\mathbb{RP}^n$ to $\mathbb{CP}^n$ and it all still works nicely (forms smooth 2n-manifold).

Example 1.22: Take $X = \mathbb{R} \times \{1,2\}/\sim$ where $(x,1) \sim (x,2)$ if $x < 0$.

Pseudo atlas given by $\mathbb{R} \times \{i\} \xrightarrow{\sim} \mathbb{R}$. But X is <u>not</u> Hausdorff.



Remark 1.23: Need not start with a set X, but could start with $\{V_\alpha\}$ in $\mathbb{R}^n$ and specify how to glue them in some smooth way.

1.3. Smooth maps

Fix manifolds X, Y with atlases $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in A}$ and $\{\psi_\beta : S_\beta \to T_\beta\}_{\beta \in B}$.

Dfn 1.24 a map $F : X \to Y$ is smooth if its continuous and $\forall \alpha, \beta$,

$$\psi_\beta \circ F \circ \varphi_\alpha^{-1} : \varphi_\alpha(F^{-1}(S_\beta)) \cap V_\alpha \to T_\beta$$

is smooth as a map between open subsets of $\mathbb{R}^{\dim X}$ and $\mathbb{R}^{\dim Y}$

Rem 1.25: We ask F to be continuous so that $\varphi_\alpha(F^{-1}(S_\beta))$ is open, so that smoothness makes sense.

Example 1.26
(i) id_X is smooth
(ii) Any constant map is smooth.
(iii) The projections $pr_1 : X \times Y \to X$ and $pr_2 : X \times Y \to Y$ are smooth
(iv) The inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth.

lemma 1.27: We have the following basic properties
(i) A map $f : X \to \mathbb{R}$ is smooth iff its smooth in the sense of 1.1
(ii) a map between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ is smooth iff its smooth in the multivariable calculus sense
(iii) Smoothness is local in the source: its enough to check it locally near each $p \in X$.
(iv) a composition of smooth maps is smooth.

Example 1.28: Viewing $\mathbb{C}^{n+1}$ as $\mathbb{R}^{2(n+1)}$, can think of S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$. Any point $x \in S^{2n+1}$ then defines a point $\mathbb{C}x \in \mathbb{CP}^n$. This gives a map $H : S^{2n+1} \to \mathbb{CP}^n$ called the Hopf map. This is smooth (Ex. sheet 1)

Dfn 1.29: A diffeomorphism $X \to Y$ is a smooth map with a smooth two-sided inverse.

Exm 1.30: $\mathbb{CP}^1$ is diffeo. to S^2. So it makes sense to think of $\mathbb{CP}^1$ as a sphere - the Riemann sphere

Lem 1.31: If X, Y are diffeomorphic, non-empty manifolds, $\dim X = \dim Y$.

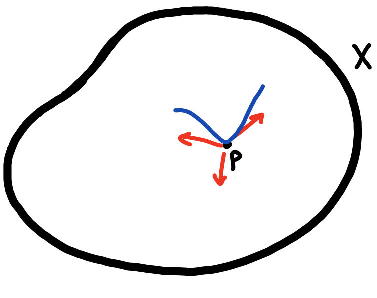
pf: pick a point $p \in X$, and a diffeo $F : X \to Y$. Pick charts $\varphi : U \to V$ about p, $\psi : S \to T$ about $F(p)$. By shrinking charts, wlog $F(U) = S$.

$$\begin{tikzcd}
 & U \arrow[r, "F"] \arrow[d, "\varphi"'] & S \arrow[d, "\psi"] & \\
\mathbb{R}^{\dim X} \supset & V \arrow[r, shift left, "G"] & T \arrow[l, shift left, "H"] & \subset \mathbb{R}^{\dim Y}
\end{tikzcd}$$

Let $G = \psi \circ F \circ \varphi^{-1}$, $H = \varphi \circ F^{-1} \circ \psi^{-1}$. Then G and H are mutually inverse smooth maps between open subsets $V \subset \mathbb{R}^{\dim X}$ and $T \subset \mathbb{R}^{\dim Y}$.

Then $D_{\varphi(p)} G$, $D_{\psi(F(p))} H$ (in usual multivariable calculus sense) are mutually inverse linear maps $\mathbb{R}^{\dim X} \rightleftarrows \mathbb{R}^{\dim Y}$, $\Rightarrow \dim X = \dim Y$.

1.4 Tangent Spaces



Fix an n-manifold X and a point $p \in X$.

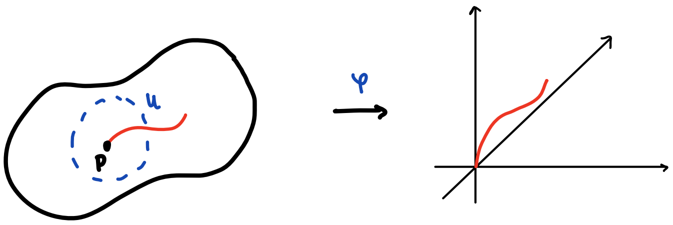
Dfn 1.32: a **curve based at p** is a smooth map $\gamma : I \to X$, I = some open n-hood of $0 \in \mathbb{R}$ such that $\gamma(0) = p$. We say two curves γ_1, γ_2 agree to first order at p if there exists a chart $\varphi : U \to V$ about p such that

$$(\varphi \circ \gamma_1)^{\cdot}(0) = (\varphi \circ \gamma_2)^{\cdot}(0) \qquad (*)$$

time derivative (wrt $t \in I$)

as vectors in $\mathbb{R}^n$.

Idea:



wlog $\varphi(p) = 0$

Lemma 1.33: If (*) holds for some chart φ about p, then it holds for all such charts about p.

pf: given a chart φ about p, write π_p^{φ} for the map

$$\pi_p^{\varphi} : \{\text{curves based at } p\} \to \mathbb{R}^n$$
$$\gamma \longmapsto (\varphi \circ \gamma)'(0).$$

A can be represented by the Jacobian of $\varphi_2 \circ \varphi_1^{-1}$ at $\varphi_1(p)$, so composition and multiplication are equiv.

Now suppose φ_1, φ_2 are two different charts about p. Then by the chain rule, $\pi_p^{\varphi_2} = A \circ \pi_p^{\varphi_1}$, where A is the derivative of $\varphi_2 \circ \varphi_1^{-1}$ at $\varphi_1(p)$.

↳ by dfn of smooth atlas, $\varphi_2 \circ \varphi_1^{-1}$ is smooth and so Jacobian determinant is nonzero.

Note A is invertible. So for curves γ_1, γ_2, we have

$$\pi_p^{\varphi_2}(\gamma_1) = \pi_p^{\gamma_2}(\gamma_2) \Leftrightarrow \pi_p^{\varphi_1}(\gamma_1) = \pi_p^{\varphi_1}(\gamma_2)$$

Cor 1.34: Agreement to the first order is an equivalence relation on curves based at p.

Dfn 1.35: The tangent space to X at p is denoted T_pX, is

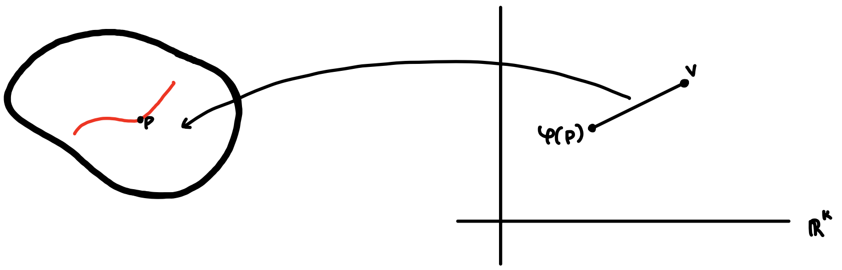
$$\{\text{curves based at } p\} \big/ \text{agreement to first order}$$

We'll write $[\gamma]$ for the tangent vector represented by γ.

Prop 1.36: T_pX naturally carries the structure of an n-dimensional vector space

pf: for each chart φ about p, π_p^{φ} induces a map $T_pX \to \mathbb{R}^n$. This is tautologically injective. We claim its surjective. If so, then π_p^{φ} will identify T_pX with $\mathbb{R}^n$, and the identifications for different φ differ by a linear automorphism of $\mathbb{R}^n$: the map A from above. So the induces vector space structure on T_pX is independent of φ.

It remains to prove π_p^{φ} is surjective. Take $v \in \mathbb{R}^n$ and consider the curve $\varphi_v : t \mapsto \varphi^{-1}(\varphi(p) + tv)$ defined on some small nhood of 0 $(-\varepsilon, \varepsilon)$. (Basically take straight line passing through $\varphi(p)$ in chart and v, and map back onto manifold)

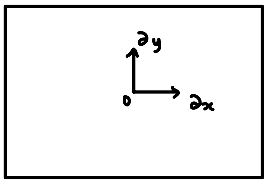


This satisfies $\pi_p^{\varphi}(\gamma_v) = v$. ☐

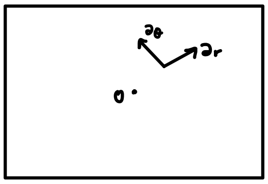
Dfn 1.37: if $x_1, \dots, x_n$ are local coordinates defined by φ and $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$, then write $\frac{\partial}{\partial x_i}, \partial x_i, \partial_i$ for the tangent vector given by $(\pi_p^\varphi)^{-1}(e_i)$.

Intuitively, ∂x_i is the direction obtained by moving along the x_i axis. I.e. keep all other x_j constant and increase x_i at unit speed.

e.g. $\mathbb{R}^2$



Cartesian coords.



polar coords

$$\frac{\partial}{\partial y_i} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial x_j}$$

idea: well,

$$\partial y_i = (\pi_p^{\varphi_2})^{-1}(e_i)$$
$$\partial y_j = (\pi p^{\varphi_1})^{-1}(e_i)$$

related by A so related

Warning: The vector ∂x_i depends on all x_j, not just x_i.

E.g. if: $y_1, \dots, y_n$ are local coords s.t $y_i = x_i$, then it need not be true that $\partial y_i = \partial x_i$.

Lemma 1.38: $\frac{\partial}{\partial y_i} = \sum_j \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial x_j}$

pf: let φ_1, φ_2 be the charts defining x, y. By definition, $\partial y_i = (\pi_p^{\varphi_2})^{-1}(e_i)$. Let $A = D(\varphi_2 \circ \varphi_1^{-1})$, so that $\pi_p^{\varphi_2} = A \circ \pi_p^{\varphi_1}$. Get $\partial y_i = (\pi_p^{\varphi_1})^{-1}(A^{-1} e_i)$. Note $A^{-1} = D(\varphi_1 \circ \varphi_2^{-1})$. So $A^{-1} e_i = \sum_j \frac{\partial x_j}{\partial y_i} e_j$. Hence

$$\partial y_i = (\pi_p^{\varphi_1})^{-1}\left(\sum_j \frac{\partial x_j}{\partial y_i} e_j\right)$$

$$= \sum_j \frac{\partial x_j}{\partial y_i} \underbrace{(\pi_p^{\varphi_1})^{-1}(e_j)}_{\partial x_j} \leftarrow (\pi_p^{\varphi_1})^{-1} \text{ is linear}$$

e.g. $A^{-1} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & & \frac{\partial x_n}{\partial y_n} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{\partial}{\partial y_1} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

☐

Rem 1.39: If $[\gamma] = \sum a_i \partial x_i$. Then $(\underbrace{\varphi \circ \gamma}_{})^{\cdot}(0) = \pi_p^{\varphi}(\gamma) = \sum a_i e_i$

i^{th} component
$(x_i \circ \gamma)^{\cdot}(0)$

Hence, $a_i = (x_i \circ \gamma)^{\cdot}(0)$.

So the coefficients of the ∂x_i are the derivatives of the x_i along γ.

> The tangent space of X at a point p is represented by the set of curves up to first order based at p. Under the map π_p^{φ}, T_pX has the structure of an $\dim X = n$-dimensional vector space. A basis of T_pX is then $\partial x_i := (\pi_p^{\varphi})^{-1}(e_i)$. Any choice of chart φ about p will do, they're all equivalent (related by A above).
>
> Equivalently: can consider them as linear maps $X_p : C^\infty(M) \to \mathbb{R}$ by the action

$$[\gamma] \in T_pX \mapsto (f \circ \gamma)^{\cdot}(0) \Leftrightarrow \frac{d}{dt}(f \circ \gamma)\Big|_{t=0}$$

Which obeys the Leibniz rule from the product rule on differentiable functions $\mathbb{R}^n \to \mathbb{R}^m$.

1.5. Derivatives

Fix manifolds X, Y and a smooth map $F: X \to Y$.

$\gamma: I \to X$, so $F \circ \gamma : I \to Y$
(based at p) (based at $F(p)$)

Dfn 1.40 : The derivative of F at p, written $D_p F$, is the map $T_p X \to T_{F(p)} Y$; $[\gamma] \mapsto [F \circ \gamma]$.
We sometimes write $D_p F$ as F_*, and call it the "push forward".

Lemma 1.41 : The map $D_p F$ is well defined and is linear.

pf: Fix a chart φ about p, ψ about $F(p)$. We have

$$\begin{aligned} \pi_{F(p)}^{\psi} (F \circ \gamma) &= (\psi \circ F \circ \gamma)^{\bullet}(0) \\ &= [(\psi \circ F \circ \varphi^{-1}) \circ (\varphi \circ \gamma)]^{\bullet}(0) \\ &= T \, \pi_p^{\varphi}(\gamma) , \end{aligned}$$

where $T = D(\psi \circ F \circ \varphi^{-1})$. So if γ_1, γ_2 are two curves based at p with $[\gamma_1] = [\gamma_2]$, then $[F \circ \gamma_1] = [F \circ \gamma_2]$. Since $\pi_p^{\varphi}(\gamma_1) = (\varphi \circ \gamma_1)^{\bullet}(0) = (\varphi \circ \gamma_2)^{\bullet}(0) = \pi_p^{\varphi}(\gamma_2)$.

So $D_p F$ is well-defined, and fits into the commutative diagram

$$\begin{tikzcd} T_p X \arrow[r, "D_p F"] \arrow[d, "\wr", "\pi_p^{\varphi}"'] & T_p Y \arrow[d, "\wr"', "\pi_{F(p)}^{\psi}"] \\ \mathbb{R}^{\dim X} \arrow[r, "T"'] & \mathbb{R}^{\dim Y} \end{tikzcd}$$

So $D_p F = (\pi_{F(p)}^{\psi})^{-1} \circ T \circ \pi_p^{\varphi}$, and hence is linear. ☐

If x, y are the local coordinates associated with φ, ψ, then $\psi \circ F \circ \varphi^{-1}$ expresses F as giving the y's in terms of the x's.

So T is $\left(\frac{\partial y_i}{\partial x_j} \right)$. Hence $D_p F(\partial_{x_i}) = \sum_j \frac{\partial y_j}{\partial x_i} \partial_{y_j}$

Rem 1.42 (i) The new notion of derivative coincides with the usual one for maps $F: \mathbb{R}^m \to \mathbb{R}^n$.
(ii) If f is a function $X \to \mathbb{R}$, then $Df(\partial_{x_i}) = \frac{\partial f}{\partial x_i}$. standard coordinate on $\mathbb{R}$.
(iii) For a curve γ based at p, we can write $[\gamma]$ as $D_0 \gamma (\partial_t)$.

Prop 1.43: For smooth maps $X \xrightarrow{F} Y \xrightarrow{G} Z$, we have

$$D_p (G \circ F) = D_{F(p)} G \circ D_p F$$

pf: For $[\gamma]$ in $T_p X$, both sides give $[G \circ F \circ \gamma]$. ☐
Remember again, $D_q(G)$ and $D_r(F)$ are maps between vector spaces.

2 VECTOR BUNDLES AND TENSORS

2.1 The tangent Bundle

Given local coordinates $x_1, \dots, x_n$ on an open set $U \subset X$, write $a_1, \dots, a_n$ for the components of a tangent vector with respect to $\partial_{x_1}, \dots, \partial_{x_n}$. This gives coordinates

$$(x_1, \dots, x_n, a_1, \dots, a_n) : \bigsqcup_{p \in U} T_p U \to \mathbb{R}^{2n}$$

Doing this for all coordinate patches U on X defines a smooth pseudo-atlas on

$$TX := \bigsqcup_{p \in U} T_p U$$

Definition 2.1 : the *tangent bundle of X is TX* equipped with the manifold structure defined by this pseudo atlas. It inherits Hausdorffness and second-countability from X.

Example 2.2 : If we think of S^1 as $\{e^{i\theta} : \theta \in \mathbb{R}\} \subset \mathbb{R}^2 = \mathbb{C}$, then although the local coordinate θ is multivalued if we try to define it globally, the vector ∂_θ is well-defined at every point. So the map

$$(p = e^{i\theta}, a\partial_\theta \in T_p S^1) \in TS^1 \mapsto (p, a) \in S^1 \times \mathbb{R}$$

is a diffeomorphism.

We'll denote a point in TX by (p, v), where $p \in X$ and $v \in T_pX$.

Definition 2.3 : A *vector field* is a smooth map $v : X \to TX$ such that $v(p)$ lies in T_pX for all p, i.e. $v : p \mapsto (p, v_p)$ for some $v_p \in T_pX$.

2.2. Vector Bundles

The tangent bundle TX of a manifold X looks like a smoothly varying family of vector spaces parametrized by X. Such families occur in many other situations.

Definition 2.4 . A *vector bundle of rank k* over a manifold B is a manifold E equipped with:

- A smooth surjection $\pi : E \to B$
- An open cover $\{U_\alpha\}_{\alpha \in A}$ of B and for each α a diffeomorphism

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$$

such that :

- $pr_1 \circ \Phi_\alpha = \pi$
- $\forall \; \alpha, \beta$, the map $\Phi_\beta \circ \Phi_\alpha^{-1}$ has the form

$$\begin{aligned} (U_\alpha \cap U_\beta) \times \mathbb{R}^k &\longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^k \\ (b, x \in \mathbb{R}^k) &\longrightarrow (b, g_{\beta\alpha}(b)(x)) \end{aligned}$$

for some smooth map $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$.

The manifold E is the total space, B is the base, π is the projection, $\pi^{-1}(p)$ are the fibres (denoted E_p), and the Φ_α are local trivializations. The $g_{\beta\alpha}$ are transition functions.

Remark 2.5: each fibre $E_p := \pi^{-1}(p)$ has the structure of a k-dimensional real vector space

Remark 2.6: really each trivialization Φ_α is like a chart, and the collection $\{\Phi_\alpha\}_{\alpha \in A}$ is like an atlas. There's an obvious notion of equivalence between two collections, and the equivalence class is what we care about.

Remark 2.7: can similarly define complex vector bundles

Example 2.8: $E = B \times \mathbb{R}^k$, $\{U_\alpha\} = \{B\}$, $\Phi : E \to B \times \mathbb{R}^k$ the obvious map, $\pi = pr_1$

This is the trivial vector bundle of rank k over B. We say Φ is a global trivialization

We denote the rank k trivial bundle by $\underline{\mathbb{R}}^k$ if the base B is clear

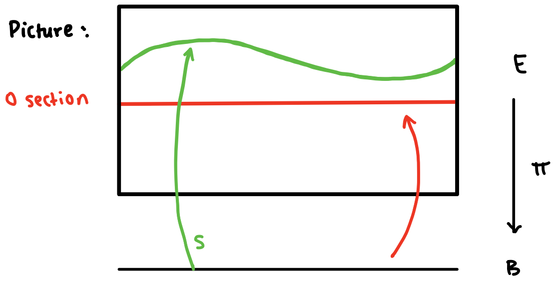
Example 2.9: TX is a rank n vector bundle, where $n = \dim X$.

Dfn 2.10: A section of a bundle $\pi : E \to B$ is a smooth map $s : B \to E$ such that $\pi \circ s = id_B$

Example 2.11: The zero section is given by $s(p) = (p, 0)$ $\forall p$.

Example 2.12: A vector field is a section of TX

Picture:



Dfn 2.13: Given a smooth map $F : B_1 \to B_2$, and $\pi_i : E_i \to B_i$, a morphism of vector bundles $E_1 \to E_2$ covering F is a smooth map $G : E_1 \to E_2$ such that

- $\pi_2 \circ G = F \circ \pi_1$
- $\forall p$, the induced map $(E_1)_p \to (E_2)_{F(p)}$ is linear

$$\begin{tikzcd} E_1 \arrow[r, "G"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "F"'] & B_2 \end{tikzcd}$$

$$\left.\begin{array}{l} \pi_2 \circ G : (p,v) \xmapsto{G} (q,w) \xmapsto{\pi_2} q \\ F \circ \pi_1 : (p,v) \xmapsto{\pi_1} p \xmapsto{F} F(p) \end{array}\right) =$$

Hence G maps fibrewise $(E_1)_p \mapsto (E_2)_{F(p)}$.

An isomorphism between vector bundles over B is a morphism covering id_B with a two-sided inverse

A bundle isomorphic to a trivial bundle is called trivial.

Example 2.14: TS^1 is trivial: $TS^1 \to S^1 \times \mathbb{R}$, $(p, a\partial_\theta) \longmapsto (p, a)$

Example 2.15: A morphism $G : \underline{\mathbb{R}} \longrightarrow E$ covering id_B is the same thing as a global section.

$$\begin{tikzcd} \underline{\mathbb{R}} \arrow[rr, "G"] \arrow[dr] & & E \arrow[dl] \\ & B & \end{tikzcd}$$

rank 1 trivial bundle

$\underline{\mathbb{R}}^k$ notation!

$G \rightsquigarrow s(p) := G(p,1)$

$s \rightsquigarrow G(p,t) := \underbrace{t\,\underbrace{s(p)}_{\in E}}_{}$

multiplication by t.

To see that $s \rightsquigarrow G(p,t) := t\,s(p)$ gives a bundle morphism: clearly the map is smooth; locally if (p,v) is an element of E, then $t\cdot(p,v) = (p, tv)$, which is smooth and respects the smooth structure on overlaps. We have that $G(p,t) := t\,s(p)$

$$\begin{tikzcd}
\underline{\mathbb{R}} \arrow[r, "G"] \arrow[d, "\mathrm{pr}_1"'] & E \arrow[d, "\pi"] \\
B \arrow[r, "\mathrm{id}"'] & B
\end{tikzcd}$$

Say $s : p \mapsto (p, \gamma(p))$, $\gamma : p \mapsto \mathbb{R}^k$, then $\mathrm{id} \circ \mathrm{pr}_1 : \underline{\mathbb{R}} \to B$; $(p,v) \mapsto p$, and $\pi \circ G : (p,t) \mapsto (p, t\gamma(p)) \mapsto p$.
So G covers id. Of course, the induced map is linear.

More generally, morphisms $\underline{\mathbb{R}}^k \to E$ correspond to k-tuples of sections. The morphism is an isomorphism iff the k-tuple forms a basis in each fibre.

Definition 2.16: Given a rank-k vector bundle E, a rank-l subbundle is a subset F of E such that $\forall p \in B$, $\exists$ a trivialization $\Phi : \pi_E^{-1}(U) \to U \times \mathbb{R}^k$ under which $\pi_F^{-1}(U)$ gets sent to $U \times (\mathbb{R}^\ell \oplus \underbrace{0}_{\text{0 vector of length } k-\ell})$.

Can then define E/F and get morphisms $F \to E \to E/F$.

2.3 Constructing Vector Bundles by Gluing

To define a vector bundle over B, its enough to give:

- A set E
- A map $\pi : E \to B$
- An open cover $\{U_\alpha\}$ of B
- For each α, a bijection $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$

Such that $\mathrm{pr}_1 \circ \Phi_\alpha = \pi$, and on overlaps $\Phi_\beta \circ \Phi_\alpha^{-1} : (p,x) \mapsto (p, g_{\beta\alpha}(p)(x))$ for some smooth $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$

Then our pseudo atlas construction makes E into a manifold (automatically Hausdorff + second-countable), and the Φ_α become trivializations.

Example 2.17: let $B = \mathbb{R}P^n = \{\text{lines in } \mathbb{R}^{n+1}\}$. Let $E = \{(p,x) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} : x \text{ lies in the line labelled by } p\}$
Define $\pi : E \to B$ by $(p,v) \to p$. Open cover $= \{U_i = \{[x_0 : \dots : x_n] : x_i \neq 0\}\}_{i=0,\dots,n}$.
Define $\Phi_\alpha : \pi^{-1}(U_i) \to U_i \times \mathbb{R}$ by

$$([x_0 : \dots : x_n], \underline{\lambda(x_0, \dots, x_n)}) \mapsto ([x_0 : \dots : x_n], \lambda x_i)$$

Check: well defined.

Then we have $\mathrm{pr}_1 \circ \Phi_i = \pi$, and $\Phi_j \circ \Phi_i^{-1}([x_0 : \dots : x_n], t) = ([x_0 : \dots : x_n], \frac{t x_j}{x_i})$

What are the transition functions? $g_{ji} : U_i \cap U_j \to GL(1, \mathbb{R}) = \mathbb{R}^*$
$[x_0 : \dots : x_n] \mapsto \frac{x_j}{x_i}$ which is smooth since $x_i, x_j \neq 0$ on $U_i \cap U_j$

This is the Tautological bundle over $\mathbb{R}P^n$. (line bundle)

In fact we can drop the set E and just specify an open cover $\{U_\alpha\}$ of B and smooth maps $g_{\beta\alpha}: U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$, such that:

- $g_{\alpha\alpha}(p) = \mathrm{id}_{\mathbb{R}^k} \quad \forall \alpha, p.$
- $\forall \alpha, \beta, \gamma, \quad g_{\gamma\alpha} = g_{\gamma\beta}\, g_{\beta\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ (cocycle condition)

Then define
$$E = \bigsqcup_\alpha U_\alpha \times \mathbb{R}^k \Big/ (p \in U_\alpha, x \in \mathbb{R}^k) \sim (p \in U_\beta, g_{\beta\alpha}(p)(x))$$

The conditions above make $\sim$ an equivalence relation.

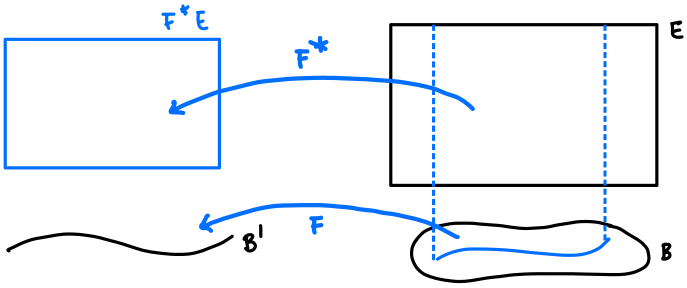
Example 2.18: For any $r \in \mathbb{Z}$, define a line (rank 1) vector bundle over $\mathbb{RP}^n$ trivialized over the U_i, where $g_{ji} = \left(\frac{x_j}{x_i}\right)^{-r}$. This is denoted $\mathcal{O}_{\mathbb{RP}^n}(r)$. The tautological bundle is $\mathcal{O}_{\mathbb{RP}^n}(-1)$.

Lemma 2.19: If $\pi : E \to B$ is a rank k vector bundle, trivialized over $\{U_\alpha\}$ with transition functions $g_{\beta\alpha}$, then its isomorphic to the output of the above construction.

Corollary 2.20: To show two bundles are isomorphic, it suffices to find trivializations over the same open cover with the same transition functions.

Dfn 2.21: Given a bundle $\pi : E \to B$ and a smooth map $F : B' \to B$, the pullback bundle F^*E has total space $\bigsqcup_{p \in B'} E_{F(p)}$

With the following bundle structure: Suppose E is trivialized over some $\{U_\alpha\}$ of B with transition functions $g_{\beta\alpha}$, then F^*E is trivialized over $\{F^{-1}(U_\alpha)\}$ with transition functions $g_{\beta\alpha} \circ F$.



Idea: essentially transplant the fibres from the image of F in B onto the preimage points in B.

i.e. $(F^*E)_p = E_{F(p)}$.

Dfn 2.22: The dual bundle $E^\vee$ is the bundle over B whose total space is

$$\bigsqcup_{p \in B} (E_p)^\vee \leftarrow \text{take dual of fibres}$$

Trivialized over $\{U_\alpha\}$, with transition functions $(g_{\beta\alpha}{}^\vee)^{-1}$ (cf the dual representation). $\swarrow$ $(g_{\alpha\beta})^\vee = (g_{\alpha\beta}{}^T)^{-1}$

Example 2.23: If E is locally trivialized by smooth sections $s_1, \dots, s_k$ over $U \subset B$, then the fibre wise dual basis defines smooth sections $\sigma_1, \dots, \sigma_k$ of $E^\vee$ over U that trivialize it.

> Locally, each vector bundle looks like $U_\alpha \times \mathbb{R}^k$ for some chart U_α. We saw before that a bundle is trivial if $\exists$ a bundle morphism $\underline{\mathbb{R}}^k \to E$ covering the identity on B. We can think about this locally: $\exists$ a bundle morphism $U_\alpha \times \underline{\mathbb{R}}^k \to \pi^{-1}(U_\alpha)$, namely Φ_α. For $\underline{\mathbb{R}} \to E$ this is equivalent to a local section, and for $\underline{\mathbb{R}}^k \to E$ this is equivalent to a collection of k local sections. Since this is an <u>iso</u>, these k sections form a basis in each fibre.
>
> Motto: E is locally trivial iff $\exists$ a collection of local sections forming a fibrewise basis.

2.4 The cotangent Bundle

Fix some n-manifold X.

Dfn 2.24: The **cotangent bundle of X** is the dual of the tangent bundle. Standard notation: T^*X. The fibre over a point $p \in X$ is denoted $T_p^* X$, and is called the cotangent space at p.

Consider $\{\text{functions at } p\} = \{ (U, f) : U \text{ an open nhood of } p,\ f : U \to \mathbb{R} \text{ smooth}\}$

We say f_1, f_2 'agree to first order' at p if $D_p f_1 = D_p f_2$

Proposition 2.25: there's a canonical isomorphism

$$\{\text{functions at } p\}/\sim \longrightarrow T_p^* X$$

> The dual vector bundle has fibres $\cong (\mathbb{R}^n)^\vee$, i.e. $\{\text{linear maps} : \mathbb{R}^n \to \mathbb{R}\}$, $\cong \mathbb{R}^n$ via the standard pairing. So to show that we can think of $T_p^* X$ as the equivalence classes of functions that agree up to first order, we just need to show that each equivalence class defines a linear map $\{\text{curves based at } p\}/\text{first order} \to \mathbb{R}$ that is bijective onto $(\mathbb{R}^n)^\vee$. (As in all the defined linear maps as a space are in bijection with $(\mathbb{R}^n)^\vee$)

Proof: There's a pairing

$$\{\text{functions at } p\} \times \{\text{curves based at } p\} \longrightarrow \mathbb{R}\ ; \quad (f, \gamma) \mapsto (f \circ \gamma)^\cdot (0)$$

This induces a map from $\{\text{functions at } p\} \longrightarrow T_p^* X$; $f \mapsto ([\gamma] \longmapsto (f \cdot \gamma)^\cdot (0))$

> Independence of choice of representative:
>
> $$\begin{aligned} (f \circ \gamma_1)^\cdot (0) &= (f \circ \varphi^{-1} \circ \varphi \circ \gamma_1)^\cdot (0) \\ &= (f \circ \varphi)^{-1}_{\varphi(\gamma_1(0))} (\varphi \circ \gamma_1)^\cdot_0 \\ &= (f \circ \varphi)^{-1}_{\varphi(\gamma_2(0))} (\varphi \circ \gamma_2)^\cdot_0 \\ &= (f \circ \gamma_2)^\cdot (0) \end{aligned}$$

$\gamma = \sum a_i \frac{\partial}{\partial x_i}$, then

$$\begin{aligned} \theta(f)(\gamma) &= (f \circ \gamma)^\cdot (0) \\ &= \sum a_i (f \circ \partial x_i)^\cdot (0) \\ &= \sum a_i \frac{\partial f}{\partial x_i}\Big|_p. \end{aligned}$$

In coordinates, this map is

$$\theta : f \longmapsto \left(\sum a_i \partial_{x_i} \longmapsto \sum_i a_i \frac{\partial f}{\partial x_i}\Big|_p \right). \qquad (*)$$

We want to show that θ is surjective and that $\theta(f_1) = \theta(f_2) \Leftrightarrow f_1 \sim f_2$.

<u>surjective</u>: The coordinate functions themselves $x_1, \dots, x_n$ are sent to the duals of ∂x_i
I.e. $\theta(x_j) = (\sum a_i \partial x_i \mapsto a_j)$, that is $\theta(x_i)(\partial x_j) = \delta_{ij}$.

Last part: Observe that $\theta(f_1) = \theta(f_2) \Leftrightarrow \frac{\partial f_1}{\partial x_i} = \frac{\partial f_2}{\partial x_i}\Big|_p \ \forall i \Leftrightarrow D_p f_1 = D_p f_2 \Leftrightarrow f_1 \sim f_2$ $\square$

Notice that if $f : U \to \mathbb{R}$ is a smooth function, by the proposition, f defines an element of $T^* X$ for each $p \in U$.

Lemma 2.26: This defines a (smooth) section of $T^* X$ over U. We denote this by **df**.

pf: We saw in the previous proof (surjectivity) that $\theta(x_i) = dx_i$. Ie that $dx_1, \dots, dx_n$ are fibrewise dual to $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. Hence (by example 2.23), $dx_1, \dots, dx_n$ is a smooth basis of sections. By $(*)$, we get

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

$$\theta(x_j) = \left(\Sigma a_i \partial_{x_i} \longmapsto \sum_i a_i \frac{\partial x_j}{\partial x_i}\Big|_p \right)$$

$$= \left(\Sigma a_i \partial_{x_i} \longmapsto a_j \right)$$

Since dx_i are smooth and so is $\frac{\partial f}{\partial x_i}$, $\Rightarrow$ df is smooth.

$f: U \to \mathbb{R}$ is smooth, and $p \in U$, then we get an element of the cotangent space $T_p^* X$. In particular, we get an element of $T_p^* X$ for all $p \in U$. I.e. $\forall\, p \in U$, f gives rise to an assignment of a covector over each point $p \in U$. This assignment is smooth, i.e. is a section, which we denote by df. (Locally) we saw that we have a basis of local sections denoted dx^i, which are dual to the local basis ∂_{x_i} for the tangent bundle. Then $dx^j : p \longmapsto ((dx^j)_p : \Sigma a_i \partial_{x_i} \mapsto a_j)$, and hence by $*$:

$$df : p \longmapsto \left((df)_p : \Sigma a_i dx^i \longmapsto \Sigma a_i \frac{\partial f}{\partial x^i}\Big|_p \right)$$

So we can write

$$df = \sum_i \frac{\partial f}{\partial x_i} dx^i$$

Lemma 2.27: A section of $T^* X$ is called a 1-form. The 1-form df is called the differential of f.

By construction,

$df(v)$ = derivative of f in the direction of v.

$$df : \sum_i \frac{\partial f}{\partial x_i} dx_i \left(\frac{\partial}{\partial x_j} \right) = \frac{\partial f}{\partial x_j}$$

Remark 2.28: each dx_i depends only on x_i (in contrast to $\frac{\partial}{\partial x_i}$, which may depend on all x_i's).

Dfn 2.29: Given a smooth map $F: X \to Y$, the map $D_p F^\vee : T^*_{F(p)} Y \longrightarrow T_p^* X$ is called the pullback by F, denoted F^*.

Lemma 2.30: If $g: Y \to \mathbb{R}$ is a smooth function, then $F^* dg = d(g \circ F)$

pf: Given a vector $[\gamma] \in T_p X$, we have

$$\begin{aligned}(F^* dg)([\gamma]) = dg(D_p F([\gamma])) &= dg([F \circ \gamma]) \\ &= (g \circ F \circ \gamma)'(0) \\ &= ((g \circ F) \circ \gamma)'(0) \\ &= d(g \circ F)([\gamma]).\end{aligned}$$

?

$$\begin{aligned}F^*(dg)([\gamma]) &= dg(F_*[\gamma]) \\ &= dg([F \circ \gamma]) \\ &= (g \circ (F \circ \gamma))'(0) \\ &= ((g \circ F) \circ \gamma)'(0) \\ &= d(g \circ F).\end{aligned}$$

$\square$

says that d and F^* commute on functions $g: Y \to \mathbb{R}$.

My own additions

Dfn 1.3: A local frame of E over U is an ordered k-tuple $s_1, \dots, s_k$ of smooth sections of E over U so that for each $p \in U$, $s_1(p), \dots, s_k(p)$ forms a basis of E_p.

Claim: a local trivialisation of E on U is equivalent to a local frame of E on U

pf: Suppose $\{U_\alpha\}$ is a cover of B, and we have a trivialisation say (a diffeo)

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$$

Then we can define a local frame by: $s_i : U_\alpha \to \pi^{-1}(U_\alpha)$; $s_i(p) = \Phi_\alpha^{-1}(p, e_i)$, where e_i is the standard basis of $\mathbb{R}^k$. Clearly then on E_p, $\{s_i(p)\}_{i=1,\dots,k}$ acts like the standard basis.

Now suppose on U_α we have a local frame $s_1, \dots, s_k$. We want to define Φ_α. Well, for any $p \in U_\alpha$, $s_1(p), \dots, s_k(p)$ form a basis, and so in fact if $v_p \in E_p$, then $\exists !$ scalars $c_1, \dots, c_k$ s.t

$$v_p = \sum_{i=1}^{k} c_i s_i(p)$$

This is sufficient data to define Φ_α. We define $\Phi_\alpha(p, v_p) := (p, c_1, \dots, c_k)$ which is clearly a map $\pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$. It is also clearly smooth in p, since the s_i are smooth in p. In fact, its a diffeo ($\Phi_\alpha^{-1}(p, c_1, \dots, c_k) \mapsto (p, v_p)$ where $v_p = \sum_{i=1}^{k} c_i s_i(p)$, and this is bijective b.c. $s_i(p)$ forms a basis. It is a linear isomorphism on the fibres, since it's really just swapping one basis for another.

On the pullback bundle: if $f : B_1 \to B_2$, then for $\pi : E \to B_2$ a vector bundle, we define $f^*E := \{(p, e) \in B_1 \times E : f(p) = \pi(e)\}$

This is a well defined vector bundle with projection $\pi' : f^*E \to B_1$; $(p, e) \mapsto p$. The following diagram commutes:

$$\begin{tikzcd}
f^*E \arrow[r, "h"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\
B_1 \arrow[r, "f"'] & B_2
\end{tikzcd}$$

where we set $h : (p, e) \mapsto e$.

This bundle has fibres $(f^*E)_p = E_{f(p)}$.

(If we fix $p \in B_1$, then $(\pi')^{-1}(p) = \{(p, e) \in \{p\} \times E : f(p) = \pi(e)\}$

$$= \{(p, e) \in \{p\} \times E : e \in \pi^{-1}(f(p))\}$$

$$= \{(p, e) \in \{p\} \times E : e \in E_{f(p)}\}$$

$$\cong E_{f(p)}.$$

2.5 Multilinear Algebra

Fix U, V finite dimensional vector spaces over $\mathbb{K}$.

Dfn 2.31: The **tensor product** $U \otimes V$ (or $U \otimes_{\mathbb{K}} V$) is a $\mathbb{K}$-vector space generated by symbols $u \otimes v$ for $u \in U$, $v \in V$ modulo some relations:

$$(\lambda_1 u_1 + \lambda_2 u_2) \otimes v = \lambda_1 (u_1 \otimes v) + \lambda_2 (u_2 \otimes v)$$
$$u_1 \otimes (\mu_1 v_1 + \mu_2 v_2) = \mu_1 (u \otimes v_1) + \mu_2 (u \otimes v_2)$$

Lemma 2.32: if $e_1, \dots, e_m$ is a basis for U and $f_1, \dots, f_n$ is a basis for V, these elements $\{e_i \otimes f_j\}$ form a basis for $U \otimes V$. So $\dim(U \otimes V) = \dim(U)\dim(V)$.

Warning: general elements are not of the form $u \otimes v$, but rather some linear combination of $u \otimes v$'s.

Lemma 2.33: Tensor product is functorial: if $\alpha : U \to U'$ and $\beta : V \to V'$ are linear, $\exists$ an induced map $U \otimes V \to U' \otimes V'$ denoted by $\alpha \otimes \beta$, defined by

$$(\alpha \otimes \beta)(u \otimes v) = \alpha(u) \otimes \beta(v) \quad \text{and extended linearly}$$

Lemma 2.34 (Universal property of $\otimes$)

A map $U \otimes V \longrightarrow W$ is the same as a bilinear map $U \times V \to W$.

Example 2.35: Fix U, V, W. Composition defines a bilinear map

$$\underset{\substack{\uparrow \\ \text{linear maps} \\ V \to W}}{\mathcal{L}(V,W)} \times \mathcal{L}(U,V) \longrightarrow \mathcal{L}(U,W)$$

Get an induced linear map $\mathcal{L}(U,V) \otimes \mathcal{L}(V,W) \to \mathcal{L}(U,W)$; $\beta \otimes \alpha \mapsto \beta \circ \alpha$.

Now take $U = W = \mathbb{K}$. Then we get $V^\vee \otimes V \to \mathbb{K}$

This linear map is called contraction. Othe tensor factors come along for the ride.

e.g. $A \otimes V^\vee \otimes V \otimes B \to A \otimes K \otimes B = A \otimes B$

note: tensor with 1 dim space does nothing (linearity properties).

I.e. Contraction $V^\vee \otimes V \to \mathbb{K}$ is induced by $V^\vee \times V \to \mathbb{K}$, $(\theta, v) \mapsto \theta(v)$.

If $e_1, \dots, e_n$ is a basis for V, and $\varepsilon_1, \dots, \varepsilon_n$ is the dual basis, then

$$\varepsilon_i \otimes e_j \mapsto \delta_{ij}, \text{ or } \sum \lambda_{ij} \varepsilon_i \otimes \varepsilon_j \mapsto \sum_i \lambda_{ii}$$

Dfn 2.36: The tensor algebra on V is $TV := \bigoplus_{r=0}^{\infty} V^{\otimes r} = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus \cdots$

This is a $\mathbb{K}$-algebra with multiplication

$$\begin{aligned} V^{\otimes r_1} \times V^{\otimes r_2} &\longrightarrow V^{\otimes r_1 + r_2} \\ (p, q) &\longmapsto p \otimes q \end{aligned}$$

e.g. $(\lambda + v_1 \otimes v_2) \times v_3 = \lambda v_3 + v_1 \otimes v_2 \otimes v_3$

Ie. the multiplication is associative, unital and non-commutative.

The exterior algebra $\Lambda^{\cdot} V$ is the quotient of TV by the two-sided ideal generated by elements of the form $v \otimes v$.

[The smallest subspace of TV containing each $v \otimes v$ and closed under multiplication on both sides].

e.g. $v_1 \otimes v_2 \otimes v_2 \otimes v_3 \longmapsto 0$ in the quotient. This is an associative, unital algebra. Write $\Lambda^r V$ for the image of $V^{\otimes r}$ – called the r^{th} exterior power of V. This represents " Signed r-dimensional volumes inside V".

We write $\wedge$ for the product on ΛV induced by $\otimes$ on TV

e.g.

$$\begin{tikzcd} v_1 \otimes v_2 \arrow[r, mapsto] & v_1 \wedge v_2 \\ TV \arrow[u, phantom, "\cap" description] \arrow[r, mapsto] & \Lambda V \end{tikzcd}$$

note $v \wedge v = 0 \quad \forall v$.

Lemma 2.37: ΛV is graded commutative, i.e. $P \wedge Q = (-1)^{rs} Q \wedge P$ for $P \in \Lambda^r V$, $Q \in \Lambda^s V$.

pf: For $v, w \in V$, we have

$0 = (v+w) \wedge (v+w) = v \wedge v + v \wedge w + w \wedge v + w \wedge w = v \wedge w + w \wedge v$.

I.e. $v \wedge w = - w \wedge v$.

This deals with $r = s = 1$.

The general case follows by associativity:

e.g. $(v_1 \wedge v_2) \wedge (v_3 \wedge v_4 \wedge v_5)$ pick up rs minus signs.

$\square$

Terminology:

Dfn 2.38: By a multiindex I, we mean a tuple $(i_1,\dots,i_r)$ of elements in $\{1,\dots,n\}$ in strictly increasing order

e.g $I = 2,3,5$, $I = 1,3,7,8$ etc...

For a basis $e_1,\dots,e_n$ of V, write e_I for $e_{i_1}\wedge\dots\wedge e_{i_r}$. Similarly write $\varepsilon_I = \varepsilon_{i_1}\wedge\dots\wedge\varepsilon_{i_r}$ for dual basis $\varepsilon_1,\dots,\varepsilon_n$.

Lemma 2.39: The elements e_I where I ranges over multi-indices of length r, form a basis for $\Lambda^r V$. So $\dim \Lambda^r V = \binom{n}{r}$

$\square$

Lemma 2.40: There's a natural isomorphism $(\Lambda^r V)^\vee = \Lambda^r V^\vee$ induced by the pairing

$$(\Lambda^r V^\vee) \times \Lambda^r V \longmapsto \mathbb{K}$$

$$(\theta_1\wedge\dots\wedge\theta_r , v_1\wedge\dots\wedge v_r) \longmapsto \sum_{\sigma\in S_r} \operatorname{sgn}(\sigma)\, \theta_{\sigma(1)}(v_1)\dots\theta_{\sigma(r)}(v_r)$$

Note: e_I becomes dual to ε_I under this pairing.

Lemma 2.41: Λ^r is functorial, i.e. for any linear map $\alpha: V \to W$, we get an induced map $\Lambda^r V \longrightarrow \Lambda^r W$,

$$v_1\wedge\dots\wedge v_r \longmapsto \alpha(v_1)\wedge\dots\wedge\alpha(v_r)$$

E.g. $\Lambda^n V$ is 1-dimensional $(\dim V = n)$. And the induced map $\Lambda^n V \longrightarrow \Lambda^n V$ is the scalar $\det(\alpha)$.

2.6 Tensors and Forms

Just as for the dual bundle, you can upgrade functional algebraic operators from vector spaces to vector bundles.

Example 2.42 : Given vector bundles $E, F \to B$, trivialized over $\{U_\alpha\}$ with transition functions $g_{\beta\alpha}$, $h_{\beta\alpha}$ respectively, then $E \oplus F$ is the bundle over B with fibre $E_p \oplus F_p$ over p, and transition functions

$$g_{\beta\alpha} \oplus h_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL(rkE, \mathbb{R}) \times GL(rkF, \mathbb{R}). \to \underset{\begin{pmatrix} g_{\beta\alpha} & 0 \\ 0 & h_{\beta\alpha} \end{pmatrix}}{GL(rkE + rkF, \mathbb{R}).}$$

Can similarly define $E \otimes F$, $E^{\otimes r}$, $\Lambda^r E$.

Example 2.43: Given a smooth map $F: X \to Y$, DF is naturally a section of $T^*X \otimes F^*TY$.
(for each $p \in X$, we have

$$(T^*X \otimes F^*TY)_p = (T_pX)^\vee \otimes T_{F(p)}Y = \mathcal{L}(T_pX, T_{F(p)}Y) \quad \text{by sheet 2}$$

need to look at this

Dfn 2.44: A tensor (field) of type (p,q) is a section of

$$(TX)^{\otimes r} \otimes (T^*X)^{\otimes q}$$

An r-form is a section of $\Lambda^r T^*X$.

Note that this coincides with our earlier definition of a 1-form.

Example 2.45: a tensor of type $(0,0)$ is a section of $\underline{\mathbb{R}}$, i.e. a smooth function $f: X \to \mathbb{R}$ (also called scalar field).

A tensor of type $(1,0)$ is a vector field, type $(0,1)$ is a 1-form.

In coordinates $x_1, \ldots, x_n$, an r-form α looks like $\sum_I \alpha_I \, dx_I$, where α_I are smooth functions, and we sum over multiindices of length r.

We can view this as a tensor of type $(0,r)$ via

$$dx_{i_1} \wedge \cdots \wedge dx_{i_r} \longmapsto \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) \, dx_{i_{\sigma(1)}} \otimes \cdots \otimes dx_{i_{\sigma(r)}}. \qquad (*)$$

Example 2.46: on $\mathbb{R}^2$, a 2-form looks like $f \, dx \wedge dy$ for some smooth function f, and we can view this as

$$f \cdot (dx \otimes dy - dy \otimes dx)$$

Warning! Some authors divide by $r!$ in $(*)$

Alternative description: $\Lambda^r V$ generated by $v_1 \wedge \dots \wedge v_r$

modulo $v_1 \wedge \dots \wedge (\lambda v_i + \mu v_i') \wedge \dots \wedge v_r$
$= \lambda(v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_r) + \mu(v_1 \wedge \dots \wedge v_i' \wedge \dots \wedge v_r)$

?

and $v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_j \wedge \dots \wedge v_r = (-1)(v_1 \wedge \dots \wedge v_j \wedge \dots \wedge v_i \wedge \dots \wedge v_r)$

Tensor of type (p,q) = section of $(TX)^{\otimes p} \otimes (T^*X)^{\otimes q}$

smooth functions

Locally $\sum \boxed{T^{i_1,\dots,i_p}_{j_1,\dots,j_q}} \, \partial x_{i_1} \otimes \dots \otimes \partial x_{i_p} \otimes dx_{j_1} \otimes \dots \otimes dx_{j_q}$

And an r-form is $\sum_I \alpha_I \, dx_I = \sum_{i_1 < \dots < i_r} \alpha_{i_1,\dots,r} \, dx_{i_1} \wedge \dots \wedge dx_{i_r}$.

E.g. $\mathbb{R}^2$ with coordinates (x,y).

Tensors of type $(0,2)$ = $f_{11} \underbrace{dx \otimes dx}_{\mapsto 0} + f_{12} \underbrace{dx \otimes dy}_{dx \wedge dy} + f_{21} \underbrace{dy \otimes dx}_{dy \wedge dx \; = -dx \wedge dy} + f_{22} \underbrace{dy \otimes dy}_{0}$

$\downarrow$

2-form: $g \, dx \wedge dy$.

Tensor of type $(0,2)$ becomes 2-form: $(f_{12} - f_{21}) \, dx \wedge dy$

To go from r-forms to $(0,r)$-tensors, send $dx \wedge dy \mapsto (dx \otimes dy - dy \otimes dx)$

If $F: X \to Y$ is a diffeomorphism, then for any tensor T on X, there is a tensor $F_* T$ on Y of the same type, called the push-forward by F.

$(F_* T)_y$ = Image of $T_{F^{-1}(y)}$ $\cap$ $(T_{F^{-1}(y)} X)^{\otimes p} \otimes (T_{F^{-1}(y)} X)^{\otimes q}$ under $D_{F^{-1}(y)} F$ on each TX factor, and $(D_y F^{-1})^{\vee}$ on the T^*X factors.

Similarly, we can turn a tensor T on Y into a tensor F^*T on X, the pull back by F.
Can do the same with forms instead of tensors.

If $F: X \to Y$ is an arbitrary smooth map, you can no longer push forward, and can only pull back $(0,q)$ tensors or forms. So given an r-form α on Y, $F^*\alpha$ is an r-form on X.

2.7 Abstract Index Notation

A tensor of type (p,q) is written with p upstairs indices, q downstairs indices.

Example 2.47: T^a denotes a vector field , T_a denotes a 1-form. A tensor of type $(2,1)$ is written either as $T^{ab}{}_c$, $T^a{}_b{}^c$, $T_a{}^{bc}$, depending on whether we're thinking of it as a section of $TX \otimes TX \otimes T^*X$, $TX \otimes T^*X \otimes TX$, or $T^*X \otimes TX \otimes TX$.

Tensor product is expressed by concatenating .

Example 2.48: $S_a T^b$ is a tensor of type $(1,1)$ given by $S \otimes T$.

Contraction is expressed by a repeated index; one upstairs and one downstairs.

Example 2.49: $S_a T^a$ represents the 1-form S_a contracted with the vector field T.
Similarly
$S_{ab}{}^c T_d{}^b$ represents contracting the second T^*X factor in S with the TX factor in T.

The specific choice of labels for the indices doesn't matter, but for an equality to make sense, you must have the same uncontracted indices on both sides. Reordering indices corresponds to permuting the factors.

e.g. $g_{ab} = g_{ba}$

Warning: This notation is independent of any choice of basis , $T^a{}_b$ does not represent components. However, it's easy to turn them into coordinate expressions.

E.g. Write a vector field T as $T^i \frac{\partial}{\partial x^i}$, where T^i are the components of T wrt $\frac{\partial}{\partial x^i}$ (note: now writing x_i as x^i).

Similarly, $\alpha = \alpha_i dx^i$. We implicitly sum over repeated indices (one up, one down). The expressions for $\otimes$ and contraction in components look exactly like they did in abstract index notation.

3 DIFFERENTIAL FORMS

using summation convention

$dx^i = \frac{dx^i}{dy_j} dy^j$

$\alpha = \alpha_i dx^i = \alpha_i \left(\frac{dx^i}{dy_j} dy^j \right)$

$= \alpha_i \frac{dx^i}{dy_j} dy^j$

$\hookrightarrow \alpha_j \frac{dx^j}{dy_i} dy^i$

3.1 Exterior derivative.

Suppose α is a 1-form on X. In local coordinates, $\alpha = \alpha_i dx^i$. Let's try to naively differentiate. Get $\frac{\partial \alpha_i}{\partial x^j} dx^j \otimes dx^i$

In different coords y_i, we have $\alpha = \alpha_i' dy^i$, where $\alpha_i' = \frac{\partial x^j}{\partial y^i} \alpha_j$. Then the naive derivative in y coords is

$$\frac{\partial \alpha_i'}{\partial y^j} dy^j \otimes dy^i = \frac{\partial}{\partial y^j}\left(\alpha_k \frac{\partial x^k}{\partial y^i} \right) dy^j \otimes dy^i$$

$$= \frac{\partial \alpha_k}{\partial y^j} \frac{\partial x^k}{\partial y^i} dy^j \otimes dy^i + \alpha_k \frac{\partial^2 x^k}{\partial y^j \partial y^i} dy^j \otimes dy^i.$$

$\frac{\partial x^k}{\partial y^i} dx^i = dx^k$

$\frac{\partial \alpha_k}{\partial y_j} dy^j$

$\frac{\partial \alpha_k}{\partial y_j} dy^j = \frac{\partial \alpha_k}{\partial x_\ell} \frac{\partial x_\ell}{\partial y_j} dy^j$

$= \frac{\partial \alpha_k}{\partial x_\ell} dx$

$$= \frac{\partial \alpha_k}{\partial x^\ell} dx^\ell \otimes dx^k + \alpha_k \frac{\partial^2 x^k}{\partial y^j \partial y^i} dy^j \otimes dy^i$$

change of coordinates

PROBLEM: answer depends on which local coordinates we use. But the "error term" is symmetric, so we can kill it by replacing $\otimes$ with $\wedge$.

Definition 3.1: The exterior derivative of α, denoted $d\alpha$, is defined in local coordinates by

$$d\alpha = \frac{\partial \alpha_i}{\partial x^j} dx^j \wedge dx^i. \qquad (d\alpha \text{ a 2-form})$$

By the calculation we just did, this is coordinate independent.

Warning! This does not work for vector fields.

Definition 3.2: For an r-form $\alpha = \alpha_I dx^I$, its exterior derivative $\underset{\uparrow \, (r+1)\text{-form.}}{d\alpha} := \frac{\partial \alpha_I}{\partial x^j} dx^j \wedge dx^I$.

Easy to check this is also coord independent.

Lemma 3.3: d is $\mathbb{R}$-linear, and on 0-forms (functions), it agrees with the differential.

Proposition 3.4: d has the following properties:

(1) $d^2 = 0$ i.e. $d(d\alpha) = 0$.

(2) For p-form α, q form β, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ (graded Leibniz rule).

(3) $d(F^*\alpha) = F^*(d\alpha)$ for any smooth map $F: X \to Y$, $\alpha \in \Omega^r(Y)$

proof: 1) Take $\alpha = \alpha_I \, dx^I$ locally. Then have $d^2\alpha$:

$$d\left(\frac{\partial \alpha_I}{\partial x_j} dx^j \wedge dx^I \right)$$

$$= \frac{\partial^2 \alpha_I}{\partial x^k \partial x^j} dx^k \wedge dx^j \wedge dx^I = 0 \text{ since } \frac{\partial^2 \alpha}{\partial x^j \partial x^k} = \frac{\partial^2 \alpha}{\partial x^k \partial x^j} \text{ and } \wedge \text{ is antisymmetric.}$$

An aside: If a 1-form $\alpha = df$, then $d\alpha = 0$. So to find a 1-form that's not the differential of a function, its enough to find one, say α, s.t $d\alpha \neq 0$.

e.g. $\alpha = x\,dy$ on $\mathbb{R}^2$.

(2) Write $\alpha = \alpha_I \, dx^I$, $\beta = \beta_J \, dx^J$

Then $d(\alpha \wedge \beta) = d(\alpha_I \beta_J \, dx^I \wedge dx^J)$

$$= \frac{\partial \alpha_I}{\partial x_K} \beta_J \, dx^K \wedge dx^I \wedge dx^J + \alpha_I \frac{\partial \beta_J}{\partial x^k} dx^k \wedge dx^I \wedge dx^J$$

$$= \left(\frac{\partial \alpha_I}{\partial x_K} dx^K \wedge dx^I \right) \wedge \left(\beta_J \, dx^J \right) + (-1)^p \left(\alpha_I \, dx^I \right) \wedge \left(\frac{\partial \beta_J}{\partial x^k} dx^k \wedge dx^J \right)$$

$$= d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

(3) Suppose $F : X \to Y$ smooth, $\alpha \in \Omega^r(Y)$.

Let $\alpha = \alpha_I \, dy^I$.

Then $d(F^*\alpha) = d(F^*(\alpha_I \, dy^{i_1} \wedge \dots \wedge dy^{i_r}))$

$$= d\left((\alpha_I \circ F) \underset{\overset{\shortparallel}{dF^*y^{i_1}}}{(F^* dy^{i_1})} \wedge \dots \wedge (F^* dy^{i_r}) \right) \text{ by an earlier result in section 2}$$

$$= d\left((\alpha_I \circ F) \, d(y^{i_1} \circ F) \wedge \dots \wedge d(y^{i_r} \circ F) \right) \qquad \text{Lemma 2.30: } F^*(df) = d(f \circ F) =: d(F^* f)$$

$$= d(\alpha_I \circ F) \wedge d(y^{i_1} \circ F) \wedge \dots \wedge d(y^{i_r} \circ F) \quad \text{using Leibniz + } d^2 = 0. \text{ ((i) and (ii)).}$$

$$= F^* d\alpha_I \wedge F^* dy^{i_1} \wedge \dots \wedge F^* dy^{i_r}$$

$$= F^* d(\alpha_I \, dy^{i_1} \wedge \dots \wedge dy^{i_r}) \quad \text{by section 2}$$

$$= F^*(d\alpha)$$

$\square$

In fact, these three properties uniquely determine d among all $\mathbb{R}$-linear maps $\Omega^\bullet(X) \to \Omega^{\bullet+1}(X)$ that coincide with d on $\Omega^0(X)$.

An r-form α is

- **closed** if $d\alpha = 0$
- **exact** if $\exists\, \beta$ s.t $\alpha = d\beta$

Rem: by (i) above, exact forms are closed.

3.2 de Rham Cohomology

Fix an n-manifold X, and write
$$Z^r(X) = \{\text{closed } r\text{-forms}\}$$
$$B^r(X) = \{\text{exact } r\text{-forms}\}$$

We saw that $B^r(X) \subseteq Z^r(X)$ since $d^2 = 0$.

Definition 3.5: The r^{th} de Rham cohomology group of X is

$$H^r_{dR}(X) = Z^r(X) / B^r(X)$$

an $\mathbb{R}$-vector space. Note $H^r_{dR}(X) = 0$ for $r > \dim(X)$. By definition, $H^r_{dR}(X) = 0$ for $r < 0$.

Example 3.6: We have $H^0_{dR}(X) = Z^0(X) / B^0(X)$

$$= \{\text{functions } f: X \to \mathbb{R} \text{ satisfying } df = 0\} / \underbrace{d\{(-1)\text{-forms}\}}_{} = 0$$
$$= \{\text{functions } f: X \to \mathbb{R} \text{ satisfying } df = 0\}$$
$$= \{\text{locally constant functions}\}$$
$$\cong \{\text{space of functions on connected components of } X\}$$
$$= \mathbb{R}^{\{\text{connected components of } X\}}$$

$df = \sum_i \frac{df}{dx^i} dx^i = 0$

+ linear independence

$\Rightarrow \frac{df}{dx^i} = 0 \;\; \forall i$

$\Rightarrow$ f is locally constant.

So $\dim H^0_{dR}(X) = \#$ connected components

$H^0_{dR}(X) = \mathbb{R}^c$, where $c = \#$ of connected components.

Example 3.7: We have $H^r_{dR}(pt) = 0$ unless $r = 0$ (since $\dim(pt) = 0$).
By previous example, $H^0_{dR}(pt) \cong \mathbb{R}$.

For a closed form α, write $[\alpha]$ for its class in $H^r_{dR}(X)$: the "cohomology class" of α.
We say α and β are cohomologous if $[\alpha] = [\beta]$.

Example 3.8: We know

$$H^r_{dR}(S^1) = \begin{cases} 0 & \text{if } r \neq 0,1 \\ \mathbb{R} & \text{if } r = 0 \\ ? & \text{if } r = 1 \end{cases}$$

We have $H^1 = Z^1/B^1$

$$= \{\text{all 1-forms on } S^1\} / \{\text{differentials}\}$$

A general 1-form α on S^1 looks like $f(\theta)\,d\theta$ (obviously closed since $d(f(\theta)d\theta) = \frac{\partial f}{\partial \theta}(\theta)\,d\theta \wedge d\theta = 0$), whilst a general differential looks like $\frac{\partial g}{\partial \theta}(\theta)\,d\theta$ (where f, g are 2π-periodic functions).

Note that $\int_0^{2\pi} \frac{\partial g}{\partial \theta}\,d\theta = 0$ by FTC (2π-periodic, so $g(2\pi) - g(0) = 0$). This means that the map $\Omega^1(S^1) \longrightarrow \mathbb{R}$, $f(\theta)\,d\theta \mapsto \int_0^{2\pi} f(\theta)\,d\theta$ induces a well-defined map

$$I : H^1_{dR}(S^1) \rightarrow \mathbb{R}.$$

This map is obviously linear, and surjective (take $f = 1$)

$g(\theta) = \int_0^{\theta} f(t)\,dt$

$\frac{\partial g}{\partial \theta} = \frac{\partial}{\partial \theta} \int_0^{\theta} f(t)\,dt = \frac{\partial}{\partial \theta}(F(\theta) - F(0))$

where F is the antiderivative of f

$= f(\theta) - 0 = f(\theta).$

Claim: I is an isomorphism.

Pf: Just need to prove injectivity. So suppose $I(f\,d\theta) = 0$. We want to find some g such that $f = \frac{\partial g}{\partial \theta}$. Define $g(\theta) = \int_0^{\theta} f(t)\,dt$. This g is 2π-periodic since $I(f\,d\theta) = 0$.

$\square$

Lemma 3.9: (Contravariant Functoriality).

If $F : X \rightarrow Y$ is a smooth map, then $F^* : \Omega^r(Y) \longrightarrow \Omega^r(X)$ induces a map $F^* : H^r_{dR}(Y) \rightarrow H^r_{dR}(X)$.

Pf: We need to show that if $\alpha \in Z^r(Y)$, then $F^*\alpha$ is closed, and if $\alpha' = \alpha + d\beta$, then $[F^*\alpha'] = [F^*\alpha]$, i.e. $F^*\alpha' - F^*\alpha$ is exact. These follow from F^* commuting with d:

- if $d\alpha = 0$, then $dF^*\alpha = F^* d\alpha = 0$
- $F^*(d\beta) = d(F^*\beta)$ so $F^*(d\beta)$ is also exact.

$\square$

Lemma 3.10: Wedge product of forms induces a product on $H^{\bullet}_{dR}(X)$. This is associative, graded commutative, and unital (constant function 1)

Pf: Suppose α, β are closed. Then $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta = 0$. So $\alpha \wedge \beta$ is closed.

represents a cohomology class in $H^{|\alpha|+|\beta|}_{dR}(X)$.

Also

$$\begin{aligned}(\alpha + d\gamma) \wedge (\beta + d\delta) &= \alpha \wedge \beta + d\gamma \wedge \beta + \alpha \wedge d\delta + d\gamma \wedge d\delta \\ &= \alpha \wedge \beta + d(\gamma \wedge \beta) + (-1)^{\alpha} d(\alpha \wedge \delta) + d(\gamma \wedge d\delta)\end{aligned}$$

So $(\alpha + d\gamma) \wedge (\beta + d\delta)$ is cohomologous to $\alpha \wedge \beta$. (well defined, i.e. $[\alpha] \wedge [\beta]$ is defined independent of choice of representative)

$\square$

Proposition 3.11: If $F_0, F_1 : X \to Y$ are (smoothly) homotopic, then they induce the same map $H^{\bullet}_{dR}(Y) \to H^{\bullet}_{dR}(X)$.

Say $F_0, F_1 : X \to Y$ are homotopic if $\exists$ a homotopy between them, i.e. a smooth map $F : X \times [0,1] \longrightarrow Y$ such that $F(-,0) = F_0$, $F(-,1) = F_1$.

Corollary 3.12: If $F : X \to Y$ is a homotopy equivalence ($\exists\ G : Y \to X$ s.t $G \circ F \simeq id_X$ and $F \circ G \simeq id_Y$), then F induces an isomorphism on cohomology, i.e. $F^* : H^{\bullet}_{dR}(Y) \to H^{\bullet}_{dR}(X)$ is an isomorphism.

pf: if such a G exists, then prop 3.11 says that $G^* \circ F^* = id_X$, and $F^* \circ G^* = id_Y$. So F^* is an isomorphism with inverse G^*.

$\square$

Example 3.13: (Poincaré Lemma) For all n, $H^{\bullet}_{dR}(\mathbb{R}^n) \cong H^{\bullet}_{dR}(pt)$.

3.3 Integration

We want to define $\int_X \omega$ for X an n-manifold, ω a compactly-supported n-form on X.

We need two technical ingredients: orientations and partitions of unity.

Orientations

E.g. $\int_{\mathbb{R}} f\,dx$ could mean $\int_{-\infty}^{\infty}$ or $\int_{\infty}^{-\infty}$. We need to specify which one we mean. Need an orientation on X.

Definition 3.14: an orientation of an n-dimensional real vector space V is a non-zero element of $\Lambda^n V$, modulo positive rescalings. An ordered basis $e_1, \dots, e_n$ induces an orientation $e_1 \wedge \dots \wedge e_n$. An orientation of a vector bundle $E \to B$ is a nowhere-zero section of $\Lambda^{top} E$ modulo rescaling by positive smooth functions.

We say E is orientable if it admits an orientation (equivalent to $\Lambda^{top} E$ being trivial), and it's oriented if it's equipped with a choice of orientation.

> *An aside:* $\Lambda^{TOP} E$ and nowhere vanishing sections
>
> *Claim:* Suppose $\pi : E \to B$ is a rank 1 (line) bundle. If E admits a nowhere-vanishing section, then E is trivial, i.e. $E \cong B \times \mathbb{R}$ as vector bundles.
>
> Define a vector bundle homeomorphism $F : B \times \mathbb{R} \longrightarrow E$ (also clearly smooth)
> $(p, t) \longmapsto t \cdot s(p)$
> $((p, tv)\ \forall\ (p, v) \in E)$
>
> And this is in fact a vector bundle isomorphism. Notice (1) that $F_p : \{p\} \times \mathbb{R} \to E_p$; $(p,t) \mapsto t \cdot s(p)$ is a linear isomorphism $\mathbb{R} \to E_p$, and (2) the square commutes:
> (↳ nowhere vanishing s)
>
> $$\begin{tikzcd} B \times \mathbb{R} \arrow[r, "F"] \arrow[d, "pr_1"'] & E \arrow[d, "\pi"] \\ B \arrow[r, "id"'] & B \end{tikzcd}$$

In terms of the exterior power of a vector bundle, the top one is constructed as follows. Let $g_{\beta\alpha}: U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$ are the transition functions, then $\Lambda^n E$ ($n = \text{rank } E$)

Idea: take a rank-k v.b. E, with transition functions $g_{\beta\alpha}: U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$, and we can take the wedge of fibres $\Lambda^n E_p$, and glue them together via $\Lambda^n g_{\beta\alpha}$ on the overlaps. Notice now that $\Lambda^n E_p$ is one-dim, and the map $\Lambda^n g_{\beta\alpha}$ is given explicitly as

$$(\Lambda^n g_{\beta\alpha})(e_1 \wedge \dots \wedge e_n) = \Lambda_{i=1}^{n} g_{\beta\alpha}(e_i)$$

$$= \Lambda_{i=1}^{n} \left(\sum_{j=1}^{n} g_{ji} e_j \right) \qquad \text{using } \begin{array}{l} \alpha \wedge \alpha = 0 \\ \alpha \wedge \beta = -\beta \wedge \alpha \end{array}$$

$$= \sum_{\sigma \in S_n} (-1)^{|\sigma|} \prod_{i=1}^{n} g_{i\sigma(i)} \Lambda_{i=1}^{n} e_i = \det(g_{\beta\alpha}) \, e_1 \wedge \dots \wedge e_n$$

Example 3.15: any trivial bundle is orientable. But $\mathcal{O}_{\mathbb{RP}^n}(-1)$ is non-orientable.
↳ line bundle,

Rem:
line bundle is orientable
⟺ admits nowhere vanishing section
⟺ gives v.b. iso to $B \times \mathbb{R}$
$F: (b,t) \mapsto t \cdot s(b)$.

Definition 3.16: a manifold X is oriented if it's tangent bundle TX is oriented.

Example 3.17: S^n is orientable $\forall n$ (it's the boundary of the ball).
$\mathbb{RP}^n$ is not always orientable (sheet 2).

Sending a basis for V to its dual basis induces a map $V \to V^\vee$. This induces a map $\Lambda^n V \to \Lambda^n V^\vee$, which becomes canonical after quotienting by positive rescalings. So orientations of V are equivalent to orientations of $V^\vee$.

Definition 3.18: A nowhere vanishing n-form on an n-manifold X is called a volume form. An orientation of X is equivalent to a volume form (up to positive rescaling).

Partitions of Unity

These allow us to patch together local constructions.

Definition 3.19: Given an open cover $\{U_\alpha\}$ of a manifold X, a partition of unity subordinate to this cover is a collection of smooth functions $\varphi_\alpha : X \to [0,1]$ satisfying:

- $\forall \alpha$, $\text{supp}(\varphi_\alpha) \subseteq U_\alpha$
  $$\text{supp}(\varphi_\alpha) \overset{}{=} \text{Closure}(\varphi_\alpha^{-1}(\mathbb{R}^*))$$ ← closure of space where φ_α takes nonzero values.
- $\forall p \in X$, $\exists$ open neighbourhood U of p such that all but finitely many φ_α vanish on U (local finiteness)
- $\sum_\alpha \varphi_\alpha = 1$. (constant function). Locally the sum is finite so makes sense.

Lemma 3.20: Given any open cover $\{U_\alpha\}$ of X, there exists a partition of unity subordinate to it.

proof: See Lee (Theorem 2.23). (non examinable)

Fix an oriented n-manifold X and a compactly supported n-form ω on X.

zero outside of a compact subset of X.

Definition 3.21: The integral of ω over X, denoted $\int_X \omega$, is defined as follows

- Cover X by coordinate patches $\{U_\alpha\}$ such that WLOG the local coordinates are all positively oriented (i.e. $\partial x_1 \wedge \dots \wedge \partial x_n$ coincides with the orientation on X). remember up to rescaling of positive smooth functions
- Pick a partition of unity $\{\rho_\alpha\}$ subordinate to this cover. Each $\rho_\alpha \omega$ has compact support contained in U_α. Write it in coordinates as $(\rho_\alpha \omega)_{12\dots n}\, dx_\alpha^1 \wedge \dots \wedge dx_\alpha^n$
- Define $$\int_X \omega = \sum_{\alpha \in A} \int_{\mathbb{R}^n} (\rho_\alpha \omega)_{12\dots n}\, dx^1 \dots dx^n$$ ← usual integral of a compactly supported function on $\mathbb{R}^n$.

Lemma 3.22: This is independent of choices.

pf: Suppose $\{V_\beta\}$ is another cover by coordinate patches with coords y_β, and a partition of unity $\{\sigma_\beta\}$ subordinate to this cover. We want to show that

$$\sum_\alpha \int_{\mathbb{R}^n} (\rho_\alpha \omega)_{12\dots n}\, dx^1 \dots dx^n = \sum_\beta \int_{\mathbb{R}^n} (\sigma_\beta \omega)_{12\dots n}\, dy^1 \dots dy^n$$

multiply by σ_β

not sure if this is = 1 or what?

We have

$$\begin{aligned}\sum_\alpha \int_{\mathbb{R}^n} (\rho_\alpha \omega)_{12\dots n}\, dx^1 \dots dx^n &= \sum_{\alpha,\beta} \int_{\mathbb{R}^n} (\sigma_\beta)(\rho_\alpha \omega)_{12\dots n}\, dx^1 \dots dx^n \\ &= \sum_{\alpha,\beta} \int_{\mathbb{R}^n} (\rho_\alpha)(\sigma_\beta \omega)_{12\dots n} \underbrace{\det\left(\frac{\partial y^i_\beta}{\partial x^j_\alpha}\right)_{i,j=1,\dots,n}} dx^1 \dots dx^n \\ &= \sum_{\alpha,\beta} \int_{\mathbb{R}^n} (\rho_\alpha)(\sigma_\beta \omega)_{12\dots n}\, dy^1 \dots dy^n \\ &= \sum_\beta \int_{\mathbb{R}^n} (\sigma_\beta \omega)\, dy^1 \dots dy^n \end{aligned}$$

☐

Remark 3.2

(i) All the sums involved are finite (all but finitely many terms are zero). For all $p \in \text{supp}(\omega)$, $\exists$ open set U_p containing p on which only finitely many ρ_α are nonzero. The U_p cover $\text{supp}(\omega)$, which is compact. So we can pass to a finite subcover. Hence only finitely many of the $\rho_\alpha \omega$ are nonzero.

(ii) We used orientedness of X to ensure that all Jacobians are positive.

3.4 Stokes' Theorem

The fundamental Theorem of calculus says that for a smooth function f on $[a,b]$ we have

$$\int_{[a,b]} \frac{df}{dx}\, dx = f(b) - f(a)$$

Setting $X = [a,b]$, we can write this as $\int_X df = \int_{\partial X} f$

boundary of X

Dfn 3.24: A (smooth) n-manifold-with-boundary is an ordinary n-manifold except that codomains of charts are now open subsets of $\mathbb{R}^n$ <u>or</u> $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$. [A function f on an open subset W of $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ is smooth if there exists an open set W' in $\mathbb{R}^n$ containing W such that f extends to a smooth function on W'].

Smooth maps are defined in the obvious way between manifolds-with-boundary. If X is a manifold-with-boundary, then the boundary of X, denoted ∂X, is the set of $p \in X$ s.t for some (or equivalently, all) charts, $\varphi: U \to V$ containing p, V is an open subset of $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ and $\varphi(p) \in \{0\} \times \mathbb{R}^{n-1}$.
The interior of X, denoted X° is $X \setminus \partial X$

Example 3.25: (i) An ordinary n-manifold X is an n-manifold-with-boundary, with $\partial X = \emptyset$.
(ii) The interval $[a,b]$ is a manifold-with-boundary. $\partial X = \{a,b\}$, $X^\circ = (a,b)$.

(iii) The closed unit ball $D^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is an n-manifold-with-boundary, with $D^{\circ n}$ = open unit ball, $\partial D^n = S^{n-1}$.

(iv) If X is a manifold-with-boundary and Y is a manifold then $X \times Y$ is a manifold-with-boundary. It has boundary $\partial X \times Y$.

Warning: if X, Y are MWB, then $X \times Y$ need not be a MWB. It may have corners at $\partial X \times \partial Y$.

Prop 3.26 if X is an n-MWB, then X° is an ordinary n-manifold and ∂X is an ordinary $(n-1)$-manifold.

pf: For X° it's immediate. For ∂X, for each point $p \in \partial X$, and each chart $\varphi: U \to V$ about p, define

$$\partial U = U \cap \partial X$$
$$= \varphi^{-1}\left((\{0\} \times \mathbb{R}^{n-1}) \cap V\right) \qquad \partial V = (\{0\} \times \mathbb{R}^{n-1}) \cap V$$

Then ∂U is an open nhood of p in ∂X and ∂V is open in $\{0\} \times \mathbb{R}^{n-1} \cong \mathbb{R}^{n-1}$.
And $\varphi|_{\partial U}: \partial U \to \partial V$ is a chart on ∂X about p.

□

Theorem 3.27 (Stokes' Theorem)

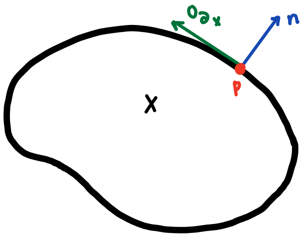
If X is an oriented n- MWB and ω is a compactly supported $(n-1)$-form on X, then

$$\int_{\partial X} \iota^* \omega = \int_X d\omega$$

$(\iota : \partial X \to X$ inclusion$)$

An aside: ∂X is oriented as follows: suppose $p \in \partial X$ and T_pX is oriented by $O_X \in \Lambda^n T_pX$. Let $n \in T_pX$ be any outward pointing normal vector. Then we orient $T_p\partial X$ by $O_{\partial X}$ defined by

$O_X = n \wedge O_{\partial X}$



Example 3.28: on $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$, oriented by $\partial x^1 \wedge \dots \wedge \partial x^n$, the vector $-\partial x^1$ is outward-pointing. So the induced orientation on $\{0\} \times \mathbb{R}^{n-1}$ is $-\partial x^2 \wedge \dots \wedge \partial x^n$.

Proof of Stokes:

Step 1: reduce to a coordinate patch. Cover X by coordinate patches $\{U_\alpha\}$ and take a partition of unity $\{\rho_\alpha\}$ subordinate to this cover.

Then $$\int_X d\omega = \int_X d\Big(\sum_\alpha \rho_\alpha \omega\Big)$$

$$= \sum_\alpha \int_{U_\alpha} d(\rho_\alpha \omega)$$

and $$\int_{\partial X} \iota^* \omega = \int_{\partial X} \iota^* \Big(\sum_\alpha \rho_\alpha \omega\Big) = \sum_\alpha \int_{\partial U_\alpha} \iota^* (\rho_\alpha \omega)$$

So its sufficient to prove

$$\int_{U_\alpha} d(\rho_\alpha \omega) = \int_{\partial U_\alpha} \iota^* (\rho_\alpha \omega).$$

Step 2: compute both sides.

By step 1, it suffices to prove the Theorem for $X = \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$. For a compactly-supported $(n-1)$-form ω on this half space, write

$$\omega = \sum_i \omega_i \, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$$

Then have $\iota^* \omega = \omega_1 dx^2 \wedge \dots \wedge dx^n$. on the boundary, which is $\{0\} \times \mathbb{R}^{n-1}$ (so where $x^1 = 0$)

So $$\int_{\partial X} \iota^* \omega = - \int_{\{0\} \times \mathbb{R}^{n-1}} \omega_1 \, dx^2 \dots dx^n$$ using the induced orientation on the boundary to give a minus.

Note
$$
\begin{aligned}
dw &= d\Big(\sum_i \omega_i \, dx^1\wedge\dots\wedge \widehat{dx^i}\wedge\dots\wedge dx^n\Big)\\
&= \sum_i d\omega_i \wedge dx^1\wedge\dots\wedge \widehat{dx^i}\wedge\dots\wedge dx^n\\
&= \sum_i \sum_j \frac{\partial w_i}{\partial x_j}\, dx^j\wedge dx^1\wedge\dots\wedge \widehat{dx^i}\wedge\dots\wedge dx^n\\
\text{(using } \alpha\wedge\alpha=0\text{)}\quad &= \sum_i \frac{\partial w_i}{\partial x^i}\, dx^i\wedge dx^1\wedge\dots\wedge \widehat{dx^i}\wedge\dots\wedge dx^n\\
&= \sum_i (-1)^{i-1}\frac{\partial w_i}{\partial x^i}\, dx^1\wedge\dots\wedge dx^i\wedge\dots\wedge dx^n
\end{aligned}
$$

So that
$$
\begin{aligned}
\int_X dw &= \sum_i \int_X (-1)^{i-1}\frac{\partial w_i}{\partial x^i}\, dx^1\dots dx^n\\
&= \int_X \frac{\partial w_1}{\partial x^1}\, dx^1\dots dx^n + \sum_{i\geq 2}(-1)^{i-1}\int_X \frac{\partial w_i}{\partial x^i}\, dx^1\dots dx^n\\
&= \underline{\int_{\mathbb{R}^{n-1}}\Big(\int_{\mathbb{R}_{\geq 0}} \frac{\partial w_1}{\partial x^1}\, dx^1\Big)\, dx^2\dots dx^n} + \underline{\sum_{i\geq 2}(-1)^{i-1}\int_{\mathbb{R}_{\geq 0}\times\mathbb{R}^{n-2}}\Big(\int_{\mathbb{R}} \frac{\partial w_i}{\partial x^i}\, dx^i\Big)\, dx^1\dots \widehat{dx^i}\dots dx^n}.
\end{aligned}
$$

The fundamental Theorem of calculus says that:

$$
= \underline{\int_{\mathbb{R}^{n-1}}\Big(\int_{\mathbb{R}_{\geq 0}} \frac{\partial w_1}{\partial x^1}\, dx^1\Big)\, dx^2\dots dx^n} = -\int_{\{0\}\times\mathbb{R}^{n-1}} \omega_1\, dx^2\dots dx^n = \int_{\partial X} \iota^* \omega .
$$

For the other terms,
$$
\underline{\int_{\mathbb{R}_{\geq 0}\times\mathbb{R}^{n-2}}\Big(\int_{\mathbb{R}} \frac{\partial w_i}{\partial x^i}\, dx^i\Big)\, dx^1\dots \widehat{dx^i}\dots dx^n} = 0
$$

☐

3.5 Applications of Stokes' Theorem

Corollary 3.29: **(Integration by parts)** Let X be an oriented n-manifold, and let α, β be a $(p-1)$-form and an $(n-p)$-form on X, at least one of which is compactly supported. Then

$$
\int_X d\alpha\wedge\beta = (-1)^p\int_X \alpha\wedge d\beta + \int_{\partial X}\alpha\wedge\beta
$$

proof: By Stokes, we have

$$
\int_X d(\alpha\wedge\beta) = \int_{\partial X}\alpha\wedge\beta
$$

By Leibniz rule,
$$
\int_X d(\alpha\wedge\beta) = \int_X d\alpha\wedge\beta + (-1)^{p-1}\alpha\wedge d\beta
$$

Put these together to get the result.

☐

Proposition 3.30: If X is a compact n-manifold, then

$$\int_X : \Omega^n(X) \longrightarrow \mathbb{R}$$

induces a map $H^n_{dR}(X) \longrightarrow \mathbb{R}$

proof: Suppose α, β are n-forms on X, such that $\alpha = \beta + d\gamma$ for some $(n-1)$-form γ.
Then

$$\int_X \alpha = \int_X \beta + \underbrace{\int_X d\gamma}_{=0}$$ by Stokes since X has no boundary.

$$\int_X d\gamma = \int_{\partial X} i^*\gamma = 0$$

□

Corollary 3.31: if X is a compact, oriented n-manifold then $H^n_{dR}(X) \neq 0$.

↱ representing orientation of X

proof: Let ω be a volume form on X. This is automatically closed so it defines a class $[\omega] \in H^n$, and we have $\int_X \omega > 0$. So $[\omega] \neq 0$.

↓ no $(n+1)$-forms on an n-manifold

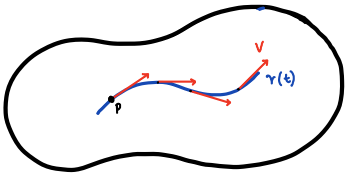
↑ nowhere vanishing

↖ if $[\omega] = [0]$, then

$$\int_X \omega = \int_X 0 =$$

□

4 FLOWS AND LIE DERIVATIVES



4.1 Flows

Fix an n-manifold X and a vector field V on X. Given a point $p \in X$, we can flow along v from p : i.e. we can try and solve the ODE for some $\gamma(t)$

$\gamma(t) \in X$, so $V(\gamma(t)) \in T_{\gamma(t)} X$ which assigns a vector $\in \mathbb{R}^n$ to the point $\gamma(t)$ in a smooth way.

$$\dot{\gamma}(t) = v(\gamma(t)) \quad , \quad \gamma(0) = p$$

By standard ODE theory, this equation has a solution defined on some $(-\varepsilon, \varepsilon)$ for $\varepsilon > 0$ sufficiently small. Moreover the solution is unique and depends smoothly on p (i.c.s). The solutions γ are called integral curves of v.

Dfn 4.1 (not - standard) a flow domain is an open neighbourhood U of $\{0\} \times X$ inside $\mathbb{R} \times X$ such that $\forall p \in X$, the set $U \cap (\mathbb{R} \times \{p\})$ is connected (i.e. is an open interval around the origin).

Dfn 4.2 A local flow of v comprises a flow domain U and a smooth map $\underset{\mathbb{R} \times X}{\overset{\cap}{\Phi : U}} \to X$ such that.

- $\Phi(0,p) = p$
- $\frac{d}{dt} \Phi(t,p) = v(\Phi(t,p)) \qquad \forall (t,p) \in U.$

fix p and you get the integral curve of v through p

It's called a global flow if $U = \mathbb{R} \times X$.

By ODEs discussion, local flows always exist and are unique in the sense that if $\Phi : U \to X$, $\Psi : V \to X$ are local flows then $\Phi = \Psi$ on $U \cap V$. We write $\Phi^t := \Phi(t, -)$.

Proposition 4.3: If Φ is the local flow of v, then $\Phi^s \circ \Phi^t = \Phi^{s+t}$ wherever this makes sense.

So in particular, $\Phi^{-t} = (\Phi^t)^{-1}$ wherever this makes sense (Φ^0 = identity by dfn).

proof: Fix $p \in X$ such that $\Phi^t(p)$, $\Phi^{s+t}(p)$ and $\Phi^s \circ \Phi^t(p)$ are defined. Let $q = \Phi^t(p)$.

Consider the curves
$$\gamma_1(\lambda) = \Phi^{\lambda s}(q)$$
$$\gamma_2(\lambda) = \Phi^{\lambda s + t}(p)$$

$$\begin{cases} \gamma_1(0) = \Phi^0(q) = id(q) = q \\ \gamma_2(0) = \Phi^{0+t}(p) = \Phi^t(p) = q \end{cases}$$

Our assumptions ensure that γ_1, γ_2 are defined on $[0,1]$. Moreover they satisfy $\gamma_1(0) = q = \gamma_2(0)$ and $\dot{\gamma}_1(\lambda) = sv(\gamma_1(\lambda))$ and $\dot{\gamma}_2(\lambda) = sv(\gamma_2(\lambda))$. So γ_1 and γ_2 are both integral curves of sv with the same initial conditions. Therefore $\gamma_1 = \gamma_2$. Hence

e.g. $\frac{d}{d\lambda} \Phi(\lambda s, p) = \frac{d}{d\lambda s} \Phi(\lambda s, p) \frac{d(\lambda s)}{d\lambda}$
$= v(\Phi(\lambda s, p)) s$
$= sv(\gamma_1(\lambda))$

$$\Phi^s \circ \Phi^t(p) = \gamma_1(1) = \gamma_2(1) = \Phi^{s+t}(p)$$

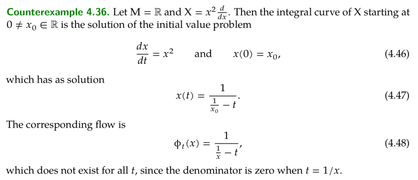
☐

Dfn 4.4 : A vector field is called complete if it admits a global flow.

Not all vector fields are complete : (e.g. $x^2 \partial_x$ on $\mathbb{R}$), but compactly supported vector fields are complete.

(Construct a local flow Φ on $(-\varepsilon, \varepsilon) \times X$, then define $\Phi^t = (\Phi^{t/N})^{\circ N}$ for $N \gg 0$. This is well defined by prop 4.3)

4.2. Lie Derivative



Counterexample 4.36. Let $M = \mathbb{R}$ and $X = x^2\frac{d}{dx}$. Then the integral curve of X starting at $0 \neq x_0 \in \mathbb{R}$ is the solution of the initial value problem

$$\frac{dx}{dt} = x^2 \quad \text{and} \quad x(0) = x_0, \tag{4.46}$$

which has as solution

$$x(t) = \frac{1}{\frac{1}{x_0} - t}. \tag{4.47}$$

The corresponding flow is

$$\phi_t(x) = \frac{1}{\frac{1}{x} - t}, \tag{4.48}$$

which does not exist for all t, since the denominator is zero when $t = 1/x$.

Fix manifold X, and vector field v. Let Φ be a local flow of v.

wlog flow domain $(-\varepsilon, \varepsilon) \times X$

$\Phi : (-\varepsilon,\varepsilon) \times X \to X$

Dfn 4.5: The **Lie derivative** of a tensor T on X along v is

$$\mathcal{L}_v T := \frac{d}{dt}\Big|_{t=0} (\Phi^t)^* T$$

takes tensor of type (p,q) to a tensor of type (p,q)

It measures how T changes along the flow. It's independent of choice of local flow Φ.

(i.e. independent of choice of flow domain)

↳ looking at $t=0$.

For an arbitrary t we have

$(\Phi^t)^* T$ pulls T back to a tensor of $(-\varepsilon,\varepsilon) \times X$ from one of X.

$$\frac{d}{dt}(\Phi^t)^* T = \frac{d}{dh}\Big|_{h=0} (\Phi^{t+h})^* T$$

basic fact of calculus

$$= \frac{d}{dh}\Big|_{h=0} (\Phi^t)^*(\Phi^h)^* T \leftarrow (\Phi^{t+h})^* = (\Phi^{h+t})^*$$

commutativity of +

$$= (\Phi^t)^* \frac{d}{dh}\Big|_{h=0} (\Phi^h)^* T \qquad = (\Phi^h \circ \Phi^t)^*$$

(prop 4.3)

$$= (\Phi^t)^* \circ (\Phi^h)^*$$

(contravariant)

$[\Phi^t)$

$$= (\Phi^t)^* \mathcal{L}_v T.$$

Lemma 4.6: For a function f, $\mathcal{L}_v f = df(v) \in C^\infty(M)$

$df : [\gamma] \longmapsto (f \circ \gamma)^\cdot(0)$

For a vector field w, $\mathcal{L}_v w = \left(v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i}\right)$ in local coordinates.

proof: For a function f, and an arbitrary point $p \in X$, we can think of $\overset{\text{fix } p}{\Phi(t,p)} : (\varepsilon,\varepsilon) \to X$ as the curve based at p representing the vector field v at p $(v(p))$. Then we have

$$\mathcal{L}_v f = \frac{d}{dt}\Big|_{t=0} (\Phi^t)^* f$$

$$= \frac{d}{dt}\Big|_{t=0} (f \circ \Phi^t)$$

(dfn of pullback of functions)

$$= (f \circ \Phi^t)^\cdot(0)$$

Φ^t is an integral curve representing v.

$$= df(v(p)).$$

To prove the second part, let Ψ be a local flow of w. At a point p in our coordinate patch.

Then

$$\mathcal{L}_v w(p) = \frac{d}{dt}\Big|_{t=0} (\Phi^t)^* w(p)$$

Pull back of a vector field
$\Phi^* w = ((D_p \Phi)^{-1} w)(\Phi^t(p))$

- $\frac{d}{dt}\Phi(t,p) = v(\Phi(t,p))$

$$= \frac{d}{dt}\Big|_{t=0} (D_p \Phi^t)^{-1} w(\Phi^t(p))$$

$$= \frac{d}{dt}\Big|_{t=0} (D_p \Phi^t)^{-1} \frac{d}{du}\Big|_{u=0} \Psi^u \circ \Phi^t(p)$$

by definition of local flow

maybe pulling $\frac{d}{du}$ to front includes the D_p?

$$= \frac{d}{dt}\Big|_{t=0} \frac{d}{du}\Big|_{u=0} \Phi^{-t} \circ \Psi^u \circ \Phi^t(p)$$

Rem: $o(t)$ means any function $f(t)$ such that $\frac{f(t)}{t} \to 0$ as $t \to 0$

In particular, $\frac{d}{dt}\Big|_{t=0} f(t) = 0$.

$\frac{d}{du}\Psi(u,p) = \omega(\Psi(u,p))$

$\omega(\Phi^t(p))$

little o.

Let p have coordinates (x^i). Then $\Phi^t(p) = x^i + tv^i + o(t)$.

Henc $\Phi^{-t} \circ \Psi^u \circ \Phi^t (p) = \Phi^{-t} \circ \Psi^u \left(x^i + tv^i + o(t) \right)$

$$= \Phi^{-t} \circ \left(x^i + tv^i + u\left(w^i + tv^j \frac{\partial w^i}{\partial x^j} \right) + o(t) + o(u) \right) \quad ?$$

$$= x^i + tv^i + uw^i + utv^j \frac{\partial w^i}{\partial x^j} - tv^i - tuw^j \frac{\partial v^i}{\partial x_j} + o(t) + o(u)$$

Therefore : $\mathcal{L}_v w = \frac{d}{dt}\Big|_{t=0} \frac{d}{du}\Big|_{t=0} \Phi^{-t} \circ \Psi^u \circ \Phi^t (p)$

I dont really get this calc.

$$= \left(v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j} \right) \frac{\partial}{\partial x^i} .$$

Lemma 4.7

(i) For a 1-form S and vector field T,

$$\mathcal{L}_v (S_a T^a) = (\mathcal{L}_v S)_a T^a + S_a (\mathcal{L}_v T)^a$$

(ii) For any tensors S and T,

$$\mathcal{L}_v (S \otimes T) = (\mathcal{L}_v S) \otimes T + S \otimes (\mathcal{L}_v T) .$$

proof: pullback commutes with contraction and tensor product. The result then follows from the ordinary product rule:

$$\mathcal{L}_v (S \otimes T) = \frac{d}{dt}\Big|_{t=0} \left[(\Phi^t)^* (S \otimes T) \right]$$

$$= \frac{d}{dt}\Big|_{t=0} \left[(\Phi^t)^* S \otimes (\Phi^t)^* T \right] \quad \text{by functoriality of tensor}$$

$$= \left(\left[\frac{d}{dt} (\Phi^t)^* S \right] \otimes (\Phi^t)^* T + (\Phi^t)^* S \otimes \left[\frac{d}{dt} (\Phi^t)^* T \right] \right)\Big|_{t=0} \quad \text{by product rule}$$

$$= \left(\frac{d}{dt}\Big|_{t=0} (\Phi^t)^* S \right) \otimes (id)^* T + (id)^* S \otimes \left(\frac{d}{dt}\Big|_{t=0} (\Phi^t)^* T \right)$$

$$= \mathcal{L}_v S \otimes T + S \otimes \mathcal{L}_v T$$

Note : $\mathcal{L}_v w = -\mathcal{L}_w v$, which was not obvious from the definition. (see it in Lemma 4.6)

Definition 4.8: The **Lie Bracket** of two vector fields is $[v, w] := \mathcal{L}_v w = -\mathcal{L}_w v$. This makes the space of all vector fields on X into a Lie algebra: a vector space equipped with a bilinear bracket operation that is alternating $([v,v] = 0)$ and satisfies the Jacobi identity

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 .$$

Lemma 4.9: The Lie Derivative is diffeomorphism invariant, i.e. if $F : X \to Y$ is a diffeomorphism, then

$$F^*(\mathcal{L}_v T) = \mathcal{L}_{F^* v}(F^* T).$$

T a tensor on Y

v a vector field on Y

Proof: We have

$$\begin{aligned}
F^*(\mathcal{L}_v T) &= F^*\left(\frac{d}{dt}\Big|_{t=0} (\Phi^t)^* T\right) \\
&= \frac{d}{dt}\Big|_{t=0} F^*(\Phi^t)^* T \\
&= \frac{d}{dt}\Big|_{t=0} F^*(\Phi^t)^*(F^{-1})^* F^* T \\
&= \frac{d}{dt}\Big|_{t=0} (\underbrace{F^{-1}\circ\Phi^t\circ F}_{\text{flow of } F^* v})^* F^* T \\
&= \mathcal{L}_{F^* v}(F^* T).
\end{aligned}$$

pulls forward onto Y, applies Φ^t and then goes back to X.

$\square$

4.3 Homotopy Invariance of de Rham Cohomology

The Lie derivative is related to the exterior derivative

Definition 4.10: Given an r-form α and a vector field v, write $\iota_v\alpha$ or $v \lrcorner \alpha$ for the $(r-1)$-form $v^{a_1}\alpha_{a_1,\dots,a_r}$

(Contract in first entry of r-form with vector field)

Lemma 4.11 (Cartan's magic formula)

$$\mathcal{L}_v\alpha = d(\iota_v\alpha) + \iota_v(d\alpha)$$

$\square$

proof: example sheet 3.

Recall **Proposition 3.11:** if $F_0, F_1 : X \to Y$ are homotopic, then $F_0^* = F_1^*$ on dR.

proof of proposition 3.11:

proof: Suppose $F : [0,1] \times X \to Y$ is a homotopy $F_0 \simeq F_1$. Let Φ be the flow of ∂_t, i.e. translation in $[0,1]$ direction). Let i_t be the map $X \to [0,1]\times X$, $x \mapsto (t,x)$. So $i_t = \Phi^t \circ i_0$, and $F_t = F \circ i_t$.

For any r-form α on X, we have

$$F_1^*\alpha - F_0^*\alpha = \int_0^1 \frac{d}{dt} F_t^*\alpha \; dt \qquad \text{by FTC.}$$

$$= \int_0^1 \frac{d}{dt}(F \circ \Phi^t \circ i_0)^*\alpha \; dt \qquad F_t = F \circ i_t = F \circ (\Phi^t \circ i_0)$$

$$= \int_0^1 i_0^* \frac{d}{dt}(\Phi^t)^* F^*\alpha \; dt$$

$$= \int_0^1 \underbrace{i_0^*(\Phi^t)^*}_{=(i_t)^*} \mathcal{L}_{\partial_t} F^*\alpha \; dt$$

See after dfn 4.5:

$$\frac{d}{dt}(\Phi^t)^*T = \frac{d}{dh}\Big|_{h=0}(\Phi^{t+h})^*T$$

$$= \frac{d}{dh}\Big|_{h=0}(\Phi^t)^*(\Phi^h)^*T = (\Phi^t)^*\frac{d}{dh}\Big|_{h=0}(\Phi^h)^*T$$

$$= (\Phi^t)^*\mathcal{L}_v T.$$

Assume α is closed. Then by Cartan's magic formula,

$$\mathcal{L}_{\partial_t} F^*\alpha = d(\iota_{\partial_t} F^*\alpha) + \iota_{\partial_t}\underbrace{(dF^*\alpha)}_{=0 \text{ since } \alpha \text{ closed } + \ dF^* = F^*d}$$

So

$$F_1^*\alpha - F_0^*\alpha = \int_0^1 i_t^* d(\iota_{\partial_t} F^*\alpha)\, dt$$

$$= \int_0^1 \underbrace{d}_{\text{on } X}(\underbrace{i_t^*\iota_{\partial_t} F^*\alpha}_{(r-1)\text{-form on } X})\, dt$$

$$= d\left(\int_0^1 i_t^*\iota_{\partial_t} F^*\alpha \; dt\right) \quad \text{Which is exact.}$$

$$\Rightarrow [F_1^*\alpha] = [F_0^*\alpha].$$

$\square$

Rem: $\mathcal{L}_v T = \frac{d}{dt}\Big|_{t=0}(\Phi^t)^*T$ so $\frac{d}{dt}(\Phi^t)^*T = (\Phi^t)^*\mathcal{L}_v T$

$\boxed{5}$ SUBMANIFOLDS, FOLIATIONS, AND FROBENIUS INTEGRABILITY

5.1 Immersions, submersions, and local diffeomorphisms

Fix manifolds X and Y of dimension m and n, and let $F: X \to Y$ be a smooth map.

Definition 5.1: F is an immersion / submersion / local diffeomorphism (at p) if DF is injective / surjective / an isomorphism (at p). The points p at which F is a submersion are called 'regular points' of F, and all other p are called critical points. A point $q \in Y$ is a regular value if $F^{-1}(y)$ contains only regular points. Otherwise its a critical value.

The name local diffeomorphism is justified by the following:

Lemma 5.2: If D_pF is an isomorphism, then $\exists$ open neighbourhoods U of p, V of $F(p)$ such that $F|_U : U \to V$ is a diffeomorphism.

Proof: pick charts φ about p, ψ about $F(p)$. Then $g := \psi \circ F \circ \varphi^{-1}$ is a map $\mathbb{R}^n \to \mathbb{R}^m$ with invertible derivative at $\varphi(p)$. By inverse function theorem, there exist open neighbourhoods U' of $\varphi(p)$, V' of $\psi \circ F(p)$ such that g is a diffeomorphism $U' \to V'$. But this says precisely that g is a diffeomorphism $g: U \to V$, where $U := \varphi^{-1}(U)$, $V := \psi^{-1}(V')$.

$\square$

Example 5.3: Consider the map $(0,\infty) \times \mathbb{R} \to \mathbb{R}^2$; $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$. This is a local diffeo. So if we restrict the domain to $(0,\infty) \times (\theta_0, \theta_0 + 2\pi)$, then it gives a diffeo $(0,\infty) \times (\theta_0, \theta_0 + 2\pi) \to \mathbb{R}^2 \setminus \mathbb{R}_{\geq 0} \cdot (\cos\theta_0, \sin\theta_0)$. $\}$ $\mathbb{R}^2$ - the ray where map is not injective.

So (r,θ) give local coordinates on $\mathbb{R}^2 \setminus \mathbb{R}_{\geq 0} \cdot (\cos\theta_0, \sin\theta_0)$ without inverting any trig functions.

Note if $F: X \to Y$ is a local diffeo at $p \in X$, and $y_1, \dots, y_n$ are local coords about $F(p)$, then $y_1 \circ F, \dots, y_n \circ F$ are local coords about p. In these coordinates, F is the identity. Similarly if $x_1, \dots, x_n$ are local coordinates about p, then $x_1 \circ F^{-1}|_U, \dots, x_n \circ F^{-1}|_U$ give local coords about $F(p)$ in which F is the identity.

Proposition 5.4: Suppose $F: X \to Y$ is an immersion at p, and $x_1, \dots, x_n$ are coords about p. Then there exist coordinates $y_1, \dots, y_m$ about $F(p)$ such that $y \circ F = (x_1, \dots, x_n, 0, \dots, 0)$ (in these coordinates F looks like $\mathbb{R}^n = \mathbb{R}^n \oplus 0 \hookrightarrow \mathbb{R}^n \oplus \mathbb{R}^{m-n} = \mathbb{R}^m$).

Similarly, if F is a submersion at p, and $y_1, \dots, y_m$ are coordinates about $F(p)$, then $\exists$ coords $x_1, \dots, x_n$ about p in which F is a projection onto the first m components.

proof: Half of proof is on example sheet 3, and other half is similar.

$\square$

Proof of local immersion thm: let $F : X^n \to Y^m$ be an immersion at $x \in X$. Then $\exists$ local coordinates about x and $y = f(x)$ so that f looks like projection onto the first n coordinates:

$\phi(0) = x$ $\quad \varphi(0) = y$

let $\phi : U \to X$ and $\varphi : V \to Y$ be charts around x and y respectively, and by shrinking U and V if necessary let g be the map from the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[d, "\phi"'] & Y \arrow[d, "\varphi"] \\
U \arrow[r, "g"'] & V
\end{tikzcd}$$

Then F an immersion at p precisely says that $dg_0 : \mathbb{R}^n \to \mathbb{R}^m$ is injective. By a change of basis, assume dg_0 is a matrix of the form

$$\begin{bmatrix} I_n \\ \hline 0 \end{bmatrix}_{m \times n}$$

Augment g to obtain a function $G : U \times \mathbb{R}^{m-n} \to \mathbb{R}^m$; $G(x, z) \mapsto (g(x), z)$. Then dG_0 is of the form

$$\begin{pmatrix} I_n & 0 \\ 0 & I_{m-n} \end{pmatrix} = I_m$$

$\Rightarrow$ invertible $\Rightarrow$ local diffeomorphism. Now ψ and G are both local diffeos at 0, so $\psi \circ G$ is also a local diffeo at 0. shrinking nhoods $\Rightarrow$ $\psi \circ G : V \to Y$ is a local parametrization of Y near Y

if h is the canonical immersion, then $g = G \circ h$

$$\Rightarrow \quad (\psi \circ G) \circ h = \psi \circ g = f \circ \phi$$

so that f is locally equiv. to canonical immersion.

5.2 Submanifolds

Fix an n-manifold X.

Definition 5.5: A **Codimension - k Submanifold of X** is a subset $Z \subset X$ such that $\forall p \in Z$, there exist local coordinates $x_1, \dots, x_n$ about p in which Z is given by $x_1 = \dots = x_k = 0$.

Warning: This holds $\forall p \in Z$, not $\forall p \in X$.

E.g: $Z = (\mathbb{R}^2 \setminus \{0\}) \times \{0\} \subset X = \mathbb{R}^3$ is a submanifold, but near the origin its not defined by the vanishing of coordinates ($(0,0,0) \notin Z$)

Note:
- Z inherits a topology from X, which is automatically Hausdorff and second countable.
- about each $p \in Z$, we have nice coordinates $x_1, \dots, x_n$ on X. Then $x_{k+1}, \dots, x_n$ give local coords on Z
- The transition functions for these coords on Z are smooth.

Equivalent atlases on X give equivalent atlases on Z. Upshot:

Proposition 5.6: If $Z \subset X$ is a codimension - k submanifold, then its naturally a smooth $(n-k)$-manifold. Moreover, the inclusion map $\iota : Z \hookrightarrow X$ is a smooth immersion that's also a homeomorphism onto its image. And composition with ι induces a bijection

$$\{\text{Smooth maps } Y \to Z\} \xrightarrow{\iota \circ} \{\text{Smooth maps } Y \to X, \text{ with image } \subset Z\}$$

Definition 5.7: A smooth immersion that is a homeomorphism onto its image is an embedding.

Lemma 5.8: if $F : Y \to X$ is an embedding with image Z, then Z is a submanifold of X, and F induces a diffeomorphism $Y \to Z$.

Example 5.9: The inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ is an embedding. Hence S^n is a submanifold of $\mathbb{R}^{n+1}$ and the smooth structure we defined on it coincides with the submanifold smooth structure.

Finding nice coordinates is hard, but there's a much easier way to check a subset of X is a subman.

Proposition 5.10: If $F: X \to Y$ is smooth, and $q \in Y$ is a regular value, then $F^{-1}(q)$ is a submanifold of X of codim = dim Y (dim Y > dim X then $F^{-1}(q)$ is empty)

proof: Take $p \in F^{-1}(q)$ and pick local coords $y_1, \dots, y_m$ about q with $y(q) = 0$. Since q is a regular value, F is a submersion at p, so $\exists$ local coords $x_1, \dots, x_n$ about p in which F is projection

$$\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m} \twoheadrightarrow \mathbb{R}^m$$

So locally near p, $F^{-1}(q)$ is given by $\{x_1 = \dots = x_m = 0\}$. □

Example 5.11: Consider $F: \mathbb{R}^{n+1} \to \mathbb{R}$, $x \mapsto \|x\|^2$. Then $DF = 2\sum x^i dx^i$. So D_pF is surjective $\forall p \neq 0$. Hence $\forall r \in \mathbb{R}\setminus\{0\}$, the set $F^{-1}(r)$ is a codimension 1 submanifold of $\mathbb{R}^{n+1}$.

E.g. $F^{-1}(1) = S^n$ is a submanifold.

Most points $q \in Y$ are regular values:

Theorem 5.12 (Sard's Theorem): For any smooth map $F: X \to Y$, the set of critical values has measure zero in Y. More precisely, if $\varphi: U \to V$ is a chart on Y, then $\varphi(\{\text{critical values in } U\}) \subset V$ has measure zero with respect to Lebesgue measure on $\mathbb{R}^{\dim Y}$.

proof: Theorem 6.10 in Lee (2nd edition) or 2.1.18 in Nicolaescu (september 2018 version). $\square$

We'll only use the following weaker version:

Corollary 5.13: regular values are dense in Y. In particular, regular values exist.

Warning: Sard's Theorem says *nothing* about regular points.

E.g. if $\dim X < \dim Y$, then there are no regular points. So regular values $= Y \setminus F(X)$.

Definition 5.14: Submanifolds $Y, Z \subset X$ are transverse if $\forall p \in Y \cap Z$, we have $T_pY + T_pZ = T_pX$. We write $Y \pitchfork Z$.

Proposition 5.15: If $Y \pitchfork Z$ of codimension k and ℓ, then $Y \cap Z$ is a subman of codimension $k+\ell$.

proof: Fix $p \in Y \cap Z$. There exist coords $y_1, \dots, y_n$ and $z_1, \dots, z_n$ about p such that $Y = \{y_1 = \dots = y_k = 0\}$, $Z = \{z_1 = \dots = z_\ell = 0\}$. Consider the map $F: U \to \mathbb{R}^{k+\ell}$ given by $(y_1, \dots, y_k, z_1, \dots, z_\ell)$. By transversality, $T_pX \to T_pX/T_pY \oplus T_pX/T_pZ$ is surjective. So F is a submersion at p. Hence $\exists$ coords $x_1, \dots, x_n$ about p s.t $x_1 = y_1, \dots, x_k = y_k$, $x_{k+1} = z_1, \dots, x_{k+\ell} = z_\ell$. So near p, $Y \cap Z$ is given by the vanishing of $x_1, \dots, x_{k+\ell}$. So $Y \cap Z$ is a submanifold of codimension $k+\ell$. $\square$

Idea: have a map $F: U \to \mathbb{R}^{k+\ell}$, $p \mapsto (y_1(p), \dots, y_k(p), z_1(p), \dots, z_\ell(p))$.

$T_pX \to T_pX/T_pY \oplus T_pX/T_pZ$ surjective, so F is a submersion.

5.3. Frobenius Integrability

Fix an n-manifold X.

Suppose we have $D \subseteq$ a rank-k subbundle of TX. We call D a distribution. often we can specify for each $p \in M$ a linear subspace $D_p \subseteq T_pM$, and take $\bigcup_p D_p = D$. By the local frame criterion for subbundles D is a smooth dist^n iff $\forall p \in M$, $\exists$ a neighbourhood U of p on which $\exists$ smooth vector fields $X_1, \dots, X_k : U \to TM$ s.t $X_1|_p, \dots, X_k|_p$ form a basis for D_q at each $q \in U$. We say that D is (locally) spanned by the vector fields $X_1, \dots, X_k$.

Suppose $D \subseteq TX$ is a smooth dist^n. A nonempty, immersed submanifold $Y \subseteq X$ is called an integral manifold of D if $T_pY = D_p$ $\forall p \in Y$. The motivation for this chapter is investigating $\exists$ of integral manifolds when given a distribution.

Definition 5.16: A **k-plane distribution D** on X is a rank k subbundle of TX

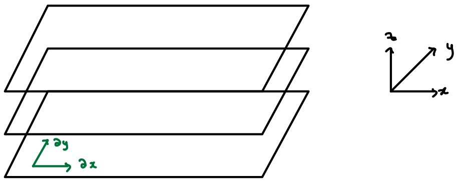
Example 5.17: In $\mathbb{R}^3$, $\langle \partial_x, \partial_y \rangle$ is a 2-plane distribution, or $\overset{\text{linear span}}{\langle \partial_x + y\partial_z, \partial_y \rangle}$. These can be described as $\ker dz$, $\ker(dz - ydx)$ respectively.

In general, a k-plane distribution is given by the vanishing of $n-k$ fibrewise linearly independent 1 forms

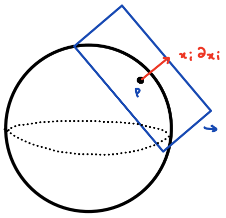
Examples 19.1: (Distributions and integral manifolds)

(a) if V is a nowhere-vanishing smooth vector field on a manifold M, then V spans a smooth rank-1 dist^n on M. The image of any integral curve of V is thus an integral manifold of D. T_pY is spanned by $\gamma'(t_0)$, where $\gamma(t_0) = p$.
1 dim^l. which is $V(\gamma(t_0))$ by dfn of integral curve.

(b) In $\mathbb{R}^n$, the vector fields $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}$ span a smooth dist^n of rank k. The k-dimensional affine subspaces parallel to $\mathbb{R}^k$ are integral manifolds.



(c) Let R be the dist^n on $\mathbb{R}^n \setminus \{0\}$ spanned by the unit radial vector field $x^i \frac{\partial}{\partial x^i}$, and let $R^\perp$ be its orthogonal complement bundle. Then $R^\perp$ is a smooth rank-$(n-1)$ dist^n on $\mathbb{R}^n \setminus \{0\}$. Through each point $x \in \mathbb{R}^n \setminus \{0\}$, the sphere of radius $|x|$ around 0 is an integral manifold of $R^\perp$.

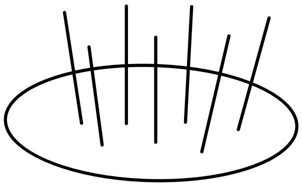


→ $T_p S^{n-1}$ is spanned by all the vectors perpendicular to $(x_i \partial x_i)_p$ (which points radially outwards

Given a k-plane distribution D, and an immersed curve γ on X (derivative of $\gamma \neq 0$), you can ask that $\dot{\gamma}$ lies in D everywhere. This is a system of $n-k$ ODE's: if D is locally $\ker(\alpha_1, \dots, \alpha_{n-k})$, then ODE's are $\alpha_i(\dot{\gamma}) = 0$.

These are invariant under reparametrization of γ.

If $k=1$, then there's a unique local solution curve (modulo parametrization) through each point. We can then pick a small $(n-1)$-dimensional disk in X transverse to D. Then get local coordinates on X, $x^1, y^1, \dots, y^{n-1}$ s.t x^1 is a coordinate along the solution curves and $y^1, \dots, y^{n-1}$ are coordinates on the disc.



Then the y^i give conserved quantities (locally) along solution curves, and conversely solution curves are any curves contained (locally) in level sets of the y^i.

If $k>1$, then the system of ODE's is underdetermined. The nicest possible situation is that there exist $n-k$ (locally) conserved quantities along solution curves, and a curve solves the system of ODEs if it lies locally in level sets of these quantities.

Dfn 5.18: Such a system of ODEs is called **integrable**

We formalise the notion of local level sets as follows:

Dfn 5.19: A smooth atlas on X is **k-foliated** if transition functions have the form locally

$$(x \in \mathbb{R}^k, y \in \mathbb{R}^{n-k}) \longmapsto (\overset{\mathbb{R}^k}{\overset{\cup}{\xi}}(x,y), \underbrace{\overset{\mathbb{R}^{n-K}}{\overset{\cup}{\eta}}(y)}_{\text{does not depend on } x}) \qquad \text{i.e. } \frac{\partial \eta}{\partial x_i} = 0 \ \forall i$$

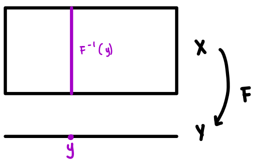
maps $\mathbb{R}^k \times \{pt\} \to \mathbb{R}^k \times \{pt'\}$

This respects the decomposition of $\mathbb{R}^n$ into slices $\mathbb{R}^k \times \{pt\}$. A k-foliation on X is an equivalence class of a foliated atlas under the obvious notion of equivalence (equivalent if their union is k-foliated).

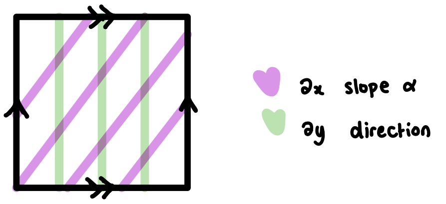
Example 5.20:

(i) if $X = Y \times Z$, then X is $\dim(Y)$ foliated by slices $Y \times \{pt\}$ (take product charts), similarly it is $\dim Z$ foliated, with slices $\{pt\} \times Z$.

(ii) if $F: X \to Y$ is a submersion, then X is foliated by fibres:



(iii) Consider the map $\mathbb{R}^2 \to T^2 = S^1 \times S^1$; $(x,y) \mapsto (e^{ix}, e^{i(\alpha x + y)})$ with $\alpha \in \mathbb{R}$. This induces local coordinates on T^2 and these induce a foliation



Can fibrate T^2 by purple slices. If α is irrational, then each slice (leaf) is dense in T^2.

Given a k-foliation of X, there's an induced k-plane distribution $D = \langle \partial_{x^1}, \dots, \partial_{x^k} \rangle$ where x^i are coordinates on the foliated atlas. These are the tangent spaces to the slices.

Conversely, given a k-plane distribution D, it arises from a k-foliation in this way $\Leftrightarrow$ the ODE system is integrable. The foliation coordinates $y^1, \dots, y^{n-k}$ correspond to the local conserved quantities).

Theorem 5.21 (Frobenius Integrability) A k-plane distribution arises from a foliation in this way iff D is closed under the Lie Bracket. I.e if $v, w \in D$ are vector fields on X, then $[v, w] \in D$.

Dfn 5.22: Such a distribution is called integrable.

Example 5.23: Recall our 2-plane dist^{n}s on $\mathbb{R}^3$:

★

i) $\langle \partial_x, \partial_y \rangle$: arises from the 2-foliation induced by standard chart on V. Can check that its closed under $[\cdot,\cdot]$:

ii) $\langle \partial_x + y\partial_z, \partial_y \rangle$: Not closed under Lie bracket: $[\partial_y, \partial_x + y\partial_z] = \partial_z \notin D$.

Suppose f is a conserved quantity $f: \mathbb{R}^3 \to \mathbb{R}$ for the ODE system (i.e. constant in direction $\partial_x + y\partial_z$ and ∂_y).

Constant along all curves tangent to $\ker(dz = -ydx)$

Then $\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial z} = \frac{\partial f}{\partial y} = 0$

$$
\begin{aligned}
\text{So} \quad f(x,y,z) &= f(0, y, z - xy) \\
&= f(0, 0, z - xy) \quad + \text{ independent of } y \\
&= \text{constant}
\end{aligned}
$$

Proof of Frobenius

Both conditions are local, so it suffices to work in a small neighbourhood of an arbitrary point $p \in X$.

Suppose D arises from a foliation. Then locally we have coordinates $x^1, \dots, x^k, y^1, \dots, y^{n-k}$ such that $D = \langle \partial_{x^1}, \dots, \partial_{x^k} \rangle$. Then from our formula for $[\cdot,\cdot]$, D is easily seen to be closed. ★

Conversely, suppose D is closed under $[\cdot,\cdot]$. Pick arbitrary local coords $s^1, \dots, s^k, t^1, \dots, t^{n-k}$ about p. By reordering and shrinking the domain we can assume that the ∂_{t^i} are transverse to D, so for $i = 1, \dots, k$ $\exists$ (unique) smooth a_{ij} such that

$$v_i = \partial_{s^i} + \sum_j a_{ij} \partial_{t^j} \quad \text{lies in } D.$$

> Idea: for a vector space V, two subspaces $U, W \leq V$ are said to be transversal if $U + W = V$, ie. every vector in V may be written as a (possibly non unique) linear combination of vectors in U and W.
>
> If $N, M \subset X$ are submanifolds satisfying $\forall\, p \in N \cap M$
>
> $$T_pN + T_pM = T_pX,$$
>
> then N and M are said to be transverse submanifolds.
>
> If you have ∂_{t^i} and D, then ∂_{t^i} and D are said to be transverse if at every point they are transversal.

Because D is a $\underline{k}$-plane distribution, wlog we can take the last n-k coords $t^1, \dots, t^{n-k}$ to be transverse to D.

Now, any ∂_{s^i} can therefore be written as something in D + something in $\langle \partial_{t^i} \rangle_i$. I.e,

$$v_i = \partial_{s^i} + \sum_j a_{ij} \partial_{t^j}$$

lies in D for some smooth coefficients a_{ij}. For some reason there are unique, perhaps to do with the fact that $\partial_{s^1}, \dots, \partial_{s^k}, \partial_{t^1}, \dots, \partial_{t^{n-k}}$ locally span TX. Actually, I also think that its dimension reasons: this is horrendous notation, but

$$\underset{\substack{\parallel \\ n}}{\dim(TX)} = \underset{\substack{\parallel \\ k}}{\dim(D)} + \underset{\substack{\parallel \\ n-k}}{\dim(\langle \partial_{t^1}, \dots, \partial_{t^{n-k}} \rangle)} - \dim(D \cap \langle \partial_{t^1}, \dots, \partial_{t^k} \rangle)$$

$$\Rightarrow \dim(D \cap \langle \partial_{t^1}, \dots, \partial_{t^{n-k}} \rangle) = 0,$$

so really this is kind of like a direct sum and therefore the decomposition is unique.

What Jack said was: WLOG, $\langle \partial_{t^1}, \dots, \partial_{t^{n-k}} \rangle$ is transverse to D (can ensure this holds at p itself, and hence on our whole coord patch after shrinking if necessary)

D is a k-plane foliation, and the v_i's are fibrewise lin. indep. since the ∂_{s^i} are, so actually $D = \langle v_1, \dots, v_k \rangle$. Hence to prove that D arises from a k-foliation of X, it suffices to construct coordinates $x^1, \dots, x^k, y^1, \dots, y^{n-k}$ such that $\partial_{x^i} = v_i$.

WLOG p = 0 in s,t coordinates. Our whole long argument just showed that for each i, there exist unique smooth functions a_{ij} such that $v_i := \partial_{s^i} + \sum_{j=1}^{n-k} a_{ij} \partial_{t^j}$ lies in D.

Let Φ_i be a local flow of v_i. Define $F : U \to X$ where U is a small neighbourhood of 0 in $\mathbb{R}^n$ by

$$F(x^1, \dots, x^k, y^1, \dots, y^{n-k}) = \Phi_1^{x^1} \circ \dots \circ \Phi_k^{x^k}(\underbrace{s = 0, t = y}_{})$$

flow for time x^i in v^i direction $\forall i$

remember $\Phi^u : X \to X$,
and X has local coords
$s^1, \dots, s^k, t^1, \dots, t^{n-k}$.

We have $D_0 F(\partial_{x^i}) = v_i(p)$, $D_0 F(\partial_{y^i}) = \partial_{t^i}(p)$, so it takes one basis to another ⇒ an isomorphism at p = 0. So $D_0 F$ is invertible. By inverse function Thm, F defines a parametrization near p.

So now we've defined our coordinates, and what's left is to show that $\partial_{x^i} = v_i$.

Suppose that $\Phi_i^{x^i}$ commute with each other. Then we have

$$\begin{aligned}
\partial_{x_i} &= \frac{d}{dt}\Big|_{t=0} \Phi_1^{x_1} \circ \dots \circ \Phi_i^{x_i + t} \circ \dots \circ \Phi_k^{x^k}(0, y) \\
&= \frac{d}{dt}\Big|_{t=0} \Phi_i^{x_i + t} \circ \Phi_1^{x_1} \circ \dots \circ \widehat{\Phi_i^{x_i}} \circ \dots \circ \Phi_k^{x^k}(0, y) \\
&= v_i(\Phi_1^{x_1} \circ \dots \circ \widehat{\Phi_i^{x_i}} \circ \dots \circ \Phi_k^{x^k}(0, y)) \in D
\end{aligned}$$

So its sufficient to prove that $\Phi_i^{x_i}$ commute. By example sheet 3, this reduces to checking that $[v_i, v_j] = 0 \quad \forall i,j$.

We have
$$[v_i, v_j] = \sum_l \left(\frac{\partial a_{jl}}{\partial s^i} - \frac{\partial a_{il}}{\partial s^j} \right) \partial_{t^l} + \sum_{m,l} \left(a_{im} \frac{\partial a_{jl}}{\partial t^m} \partial_{t^l} - a_{jl} \frac{\partial a_{im}}{\partial t^l} \partial_{t^m} \right)$$

$$\nearrow$$

$$[\partial_{s^i}, \partial_{s^{j-1}} + \sum_l a_{jl} \partial_{t^l}]$$

We're assuming that $[v_i, v_j] \in D$, but we see that it's a linear combination of ∂_{t^i}'s, which are transverse to D. Hence $[v_i, v_j]$ must be zero.

$\square$

Theorem 5.24 (Frobenius Integrability Alternate version)

A distribution D arises from a foliation iff the annihilator of D

$$I(D) := \{ \alpha \in \Omega^*(X) : \alpha(v_1, \dots, v_r) = 0 \text{ whenever all } v_i \in D \}$$

is closed under d.

E.g. $D = \langle \partial_x, \partial_y \rangle$ has $I(D) = \Omega^\cdot(\mathbb{R}^3) \wedge dz$. So if $\alpha \in I(D)$, then $\alpha = \beta \wedge dz$ for some β. Thus $d\alpha = d\beta \wedge dz \in I(D)$. So D arises from a foliation.

$D = \langle \partial_x + y\partial_z, \partial_y \rangle$ has $I(D) = \Omega^\cdot(\mathbb{R}^3) \wedge (dz - ydx)$ which is not closed under d: e.g. $d(dz - ydx) \in I(D)$. If it were, then $d(dz - ydx) \wedge (dz - ydx) = 0$ (we'd be able to write $d(dz - ydx) = \alpha \wedge (dz - ydx)$ for some α since $d(dz - ydx) \in I(D)$. But

$$\begin{aligned} & d(dz - ydx) \wedge (dz - ydx) \\ = & (-dy \wedge dx) \wedge (dz - ydx) \\ = & -dy \wedge dx \wedge dz \\ = & dx \wedge dy \wedge dz \neq 0 \end{aligned}$$

So D does not arise from a foliation.

Proof of Alternate Version

Both conditions are local, so we can work locally near p. Then $\exists$ vector fields $v_1, \dots, v_k$ near p such that $D = \langle v_1, \dots, v_k \rangle$. Similarly there exist $n-k$ 1-forms $\alpha_1, \dots, \alpha_{n-k}$ such that $D = \ker \alpha_1 \cap \dots \cap \ker \alpha_{n-k}$.

Then $I(D) = \Omega^{\cdot}(X) \wedge \alpha_1 + \dots + \Omega^{\cdot}(X) \wedge \alpha_{n-k}$

$I(D)$ is closed under d iff $\forall i$ $d\alpha_i \in I(D)$. This holds iff $d\alpha_i(v_j, v_k) = 0$ $\forall$ j,k.

Claim: For any 1-form α and vector fields S,T,

$$d\alpha(S,T) = \iota_S\, d(\iota_T \alpha) - \iota_T\, d(\iota_S \alpha) - \iota_{[S,T]}\, \alpha$$

Applying this to $d\alpha_i(v_\ell, v_m)$, we get

$$d\alpha_i(v_\ell, v_m) = \iota_{v_\ell}\, d(\underbrace{\iota_{v_m}\alpha_i}_{\substack{0 \text{ since } v_m \in D \\ \in \ker \alpha}}) - \iota_{v_m}\, d(\underbrace{\iota_{v_\ell}\alpha_i}_{\substack{=0 \text{ since } v_\ell \in D \\ \in \ker \alpha}}) - \iota_{[v_\ell, v_m]}\alpha_i$$

$$= 0 - 0 - \iota_{[v_\ell, v_m]}\alpha_i$$

$$= -\iota_{[v_\ell, v_m]}\alpha_i$$

Hence $I(D)$ is closed under D $\Leftrightarrow$ LHS $= 0$ $\forall i, \ell, m$

$\Leftrightarrow$ RHS $= 0$ $\forall i, \ell, m$

$\Leftrightarrow$ $[v_\ell, v_m] \in \ker(\alpha_i)$ $\forall i$

$\Leftrightarrow$ $[v_\ell, v_m] \in D$.

$\Leftrightarrow$ D arises from a foliation by first version of Frobenius.

So we just need to prove the claim. We have

$$\iota_S\, d(\iota_T \alpha) = \mathcal{L}_S\,(\overbrace{\iota_T \alpha}^{\iota_T\alpha \text{ is just a function}})$$

$$= \iota_{[S,T]}\alpha + \iota_T\, \mathcal{L}_S \alpha \qquad \text{(by Leibniz)}$$

And $\mathcal{L}_S \alpha = \iota_S\, d\alpha + d(\iota_S \alpha)$ by Cartan's magic formula. Putting everything together,

$$\iota_S\, d(\iota_T \alpha) = \iota_{[S,T]}\alpha + \underbrace{\iota_T \iota_S\, d\alpha}_{d\alpha(S,T)} + \iota_T\, d(\iota_S \alpha)$$

$$\Rightarrow \quad d\alpha(S,T) = \iota_S\, d(\iota_T \alpha) - \iota_T\, d(\iota_S \alpha) - \iota_{[S,T]}\alpha$$

as required. This completes the proof.

$\square$

See paper notes. for in depth calculation.

$\boxed{6}$ LIE GROUPS AND LIE ALGEBRAS

6.1 Lie Groups

Dfn 6.1: A Lie Group is a manifold G equipped with a group structure such that multiplication $m : G \times G \to G$ and inversion $i : G \to G$ are smooth

Example 6.2: $GL(n, \mathbb{R})$

Dfn 6.3: an embedded Lie subgroup of a Lie group G is a subgroup H thats also a submanifold. The restrictions of group operations from G to H are smooth, so H inherits a Lie group structure.

Example 6.4: $SL(n,\mathbb{R})$, $O(n)$, $SO(n)$ are embedded Lie subgroups of $GL(n,\mathbb{R})$. $GL(n,\mathbb{C})$, $U(n)$, $SU(n)$ are embedded Lie subgroups of $GL(2n,\mathbb{R})$.

Dfn 6.5: Given a Lie group G, and $g \in G$, we have maps $G \to G$:

$$\left.\begin{array}{ll} L_g(h) = gh & \text{left-translation} \\ R_g(h) = hg & \text{right-translation} \\ C_g(h) = ghg^{-1} & \text{conjugation} \end{array}\right\} \text{by } g$$

These are diffeomorphisms : the inverses are $L_{g^{-1}}$, $R_{g^{-1}}$, $C_{g^{-1}}$.

Dfn 6.6: A tensor T on G is left-invariant if $L_g^* T = T \ \forall g \in G$. Similarly for right-invariant and conjugation-invariant. T is called bi-invariant if its left-invariant and right invariant.

bi-invariant $\Rightarrow$ conjugation-invariant

Lemma 6.7: For any $h \in G$, The map

$$\left\{ \begin{array}{c} \text{left-invariant tensors} \\ \text{on } G \text{ of type } (p,q) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{tensors at } h \text{ of} \\ \text{type } (p,q) \end{array} \right\}$$

given by evaluation at h is a bijection. Similarly for right-invariant.

pf: if T is left-invariant, then $\forall g \in G$ we have

$$T_g = (L_{gh^{-1}})_* T_h = (L_{hg^{-1}})^* T_h \qquad (*)$$

need to go over.

So the map $T \to T_h$ is injective. Conversely, given T_h at h, the formula $(*)$ defines a left-invariant extension of T_h to G.

Corollary 6.8: Any Lie group G is parallelisable (has trivial tangent bundle).

pf: pick a basis for T_eG. The left-invariant vector fields associated to this basis form a fibrewise basis for TG, trivialising it. $\square$

Example 6.9: for even $n \geq 2$, S^n does not admit a Lie group structure (TS^n is nontrivial). On the other hand, S^3 is parallelisable, as S^3 is diffeomorphic to $SU(2)$:

$$SU(2) = \left\{ \begin{pmatrix} u & -\bar{v} \\ \bar{v} & \bar{u} \end{pmatrix} : |u|^2 + |v|^2 = 1 \right\} = S^3 \subset \mathbb{C}^2.$$

6.2. Lie Algebras

Fix a Lie group G.

Dfn 6.10: the Lie Algebra of G, denoted $\mathfrak{g}$, is T_eG

Example 6.11: For $G = GL(n, \mathbb{R})$, we have $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) := \mathrm{Mat}_{n \times n}(\mathbb{R})$

Recall a Lie Algebra is a vector space equipped with an alternating bilinear bracket which satisfies the Jacobi identity.

Proposition 6.12: $\mathfrak{g}$ carries a natural bracket operation, making it into a Lie Algebra.

proof: To each element $\xi \in \mathfrak{g}$, there is an associated left-invariant vector field ℓ_ξ. We claim that the Lie bracket of two left-invariant vector fields is left-invariant, so we can define $[\xi, \eta]$ by

$$\ell_{[\xi,\eta]} = [\ell_\xi, \ell_\eta].$$

This inherits the Lie Algebra properties from the Lie bracket of vector fields.

It remains to prove the claim. Well, for all ξ, η and $g \in G$, we have

$$L_g^* [\ell_\xi, \ell_\eta] = [L_g^* \ell_\xi, L_g^* \ell_\eta] \quad \text{by diffeomorphism invariance of } [\cdot,\cdot].$$
$$= [\ell_\xi, \ell_\eta] \quad \text{since } \ell_\xi, \ell_\eta \text{ are left-invariant.}$$

So $[\ell_\xi, \ell_\eta]$ is left-invariant.

□

Proposition 6.13: For all $\xi \in \mathfrak{g}$, the vector field is complete.

proof: Consider ODE $\dot{\gamma} = \ell_\xi(\gamma)$ with $\gamma(0) = e$

This has a solution on $(-\varepsilon, \varepsilon)$ for some small $\varepsilon > 0$. This curve satisfies

$$\gamma(s+t) = \gamma(s)\gamma(t) \qquad \text{for small } s, t.$$

(Both sides satisfy $\frac{d}{dt} = \ell_\xi$, and start at $\gamma(s)$. Hence they're equal by uniqueness of solutions).

Now extend γ to $\mathbb{R}$ by defining $\gamma(t) = \gamma(t/N)^N$ for $N >> 0$. Now define the global flow Φ of ℓ_ξ by

$$\Phi^t(g) = g\gamma(t)$$

□

We'll write Φ_ξ for the flow of ℓ_ξ.

Dfn 6.14: The exponential map $\exp: \mathfrak{g} \longrightarrow G$ is defined by $\exp(\xi) = \Phi^1_\xi(e)$.

Lemma 6.15: We could have used right invariant vector fields instead and we'd get the same exp.

proof: Let γ_ξ be the integral curve of ℓ_ξ starting at e. So $\exp(\xi) = \gamma_\xi(1)$. It suffices to show that γ_ξ is an integral curve of the right invariant vector field r_ξ. This holds since $\forall\, t$ we have

$$\begin{aligned}\dot{\gamma}_\xi(t) &= \frac{d}{ds}\Big|_{s=0} \gamma_\xi(s+t) \\ &= \frac{d}{ds}\Big|_{s=0} \gamma_\xi(s)\gamma_\xi(t) \\ &= r_\xi(\gamma_\xi(t))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}_\xi(t) &= \frac{d}{dt} \gamma_\xi(t) \\ &= \frac{d}{ds}\Big|_{s=0} \gamma_\xi(s+t) \\ &= \underbrace{\frac{d}{ds}\Big|_{s=0} \gamma_\xi(s)}_{\xi} \gamma_\xi(t)\end{aligned}$$

translating by right $\gamma_\xi(t)$

$$\begin{aligned}&\frac{d}{ds}\Big|_{s=0} \gamma_\xi(s)\gamma_\xi(t) \\ &= (R_{\gamma_\xi(t)})_* \frac{d}{ds}\gamma_\xi(s)\Big|_{s=0} \\ &= (R_{\gamma_\xi(t)})_* \ell_\xi(\Phi_\xi(s,e))\Big|_{s=0} \\ &= (R_{\gamma_\xi(t)})_* \ell_\xi(\Phi_\xi(0,e)) \\ &= (R_{\gamma_\xi(t)})_* \ell_\xi(e) \\ &= (R_{\gamma_\xi(t)})_* r_\xi(e) \\ &= r_\xi(\gamma_\xi(t)).\end{aligned}$$

$\ell_\xi(e) = \xi = r_\xi(e)$

Lemma 6.16: exp is smooth

pf: Consider the vector field v on $\mathfrak{g} \times G$ given by $v(\xi, g) = (0, \ell_\xi(g))$. This has a smooth local flow Φ, which preserves the slices $\{\xi\} \times G$. On this slice it's the flow of ℓ_ξ. So

$$\exp(\xi) = pr_2(\Phi^1(\xi, e))$$

which is smooth.

$\square$

Example 6.17: For $A \in \mathfrak{gl}(n,\mathbb{R})$, define e^A by $I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$

This converges absolutely, uniformly on compact sets. Consider $\gamma(t) := e^{tA}$. This satisfies

$$\begin{aligned}\dot{\gamma}(t) &= A + tA^2 + \frac{1}{2!}t^2A^3 + \dots \\ &= \underset{r_A(\gamma(t))}{\underset{\|}{Ae^{tA}}} = \underset{\ell_A(\gamma(t))}{\underset{\|}{e^{tA}A}}\end{aligned}$$

Hence γ is the integral curve γ_A.

Thus $\exp(A) = \gamma(1) = e^A$.

Warning! At $0 \in \mathfrak{g}$, the derivative $D_0\exp: \mathfrak{g} \to \mathfrak{g}$ is $id_\mathfrak{g}$, so exp is a local diffeomorphism near 0. But exp need not be globally injective or surjective. E.g. for $SL(2,\mathbb{R})$ its neither.

lemma 6.18: For $\xi, \eta \in \mathfrak{g}$ we have $[\xi, \eta] = \frac{d}{dt}\Big|_{t=0} (C_{\exp(t\xi)*})\,\eta$

proof: We have

$$\begin{aligned}[\xi,\eta] &= \frac{d}{dt}\Big|_{t=0}\frac{d}{du}\Big|_{u=0} \Phi^{-t}_\xi \circ \Phi^u_\eta \circ \Phi^t_\xi(e) \\ &= \frac{d}{dt}\Big|_{t=0}\frac{d}{du}\Big|_{u=0} \exp(t\xi)\exp(u\eta)\exp(-t\xi) \\ &= \frac{d}{dt}\Big|_{t=0} (C_{\exp(t\xi)})_*\,\eta\end{aligned}$$

$\square$

Corollary 6.19: For $A,B \in \mathfrak{gl}(n,\mathbb{R})$, $[A,B] = AB - BA$.

proof: By previous lemma, $[A,B] = (e^{tA}Be^{-tA})^{\cdot}(0)$.

$\square$

Corollary 6.20: If $\xi, \eta \in \mathfrak{g}$ satisfy $[\xi, \eta] = 0$, then $\exp(\xi + \eta) = \exp(\xi)\exp(\eta)$.
So in particular, $\exp(\xi)$ and $\exp(\eta)$ commute.

proof: Define $\gamma(t) = \exp(t\xi)\exp(t\eta)$.

$(L_{\exp(t\xi)})_* \, (R_{\exp(t\eta)})_* \, \xi$

We have $\dot{\gamma}(t) = \boxed{\exp(t\xi)\,\xi\,\exp(t\eta)} + \exp(t\xi)\exp(t\eta)\,\eta$

$= \exp(t\xi)\exp(t\eta)\cdot(\xi' + \eta)$

$(C_{\exp(-t\eta)})_* \xi = \exp(-t\eta)\,\xi\,(\exp(t\eta))^{-1}$

$\Rightarrow \exp(t\xi)\exp(t\eta)\xi' = \exp(t\xi)\exp(t\eta)\ \exp(-t\eta)\,\xi\,(\exp(-t\eta))^{-1}$

Where $\xi' = (C_{\exp(-t\eta)})_* \xi$

We claim $\xi' = \xi \ \forall t$, so $\dot{\gamma}(t) = \ell_{\xi+\eta}(\gamma(t))$. Then γ solves the ODE defining $\exp(t(\xi+\eta))$ and satisfies $\gamma(0) = e$ so we're done.

At $t = 0$ we have $\xi' = \xi$. But also

$$\frac{d}{dt}\xi' = \frac{d}{dh}\Big|_{h=0} \left(C_{\exp(-(t+h)\eta)}\right)_* \xi$$

$$= -\left(C_{\exp(-t\eta)}\right)_* [\eta, \xi]$$

$$= 0 \text{ by our assumption that } [\xi, \eta] = 0.$$

□

Warning! For general ξ, η, it's not true that $\exp(\xi + \eta) = \exp(\xi)\exp(\eta)$.

6.3 Lie Group Actions

Fix a Lie group G, and a manifold X

Definition 6.21: An action $\sigma: G \times X \to X$ of G on X is smooth if the map σ is smooth.

Examples 6.22:
(i) Action of G (or embedded subgroups of G) on G by left/right translation or conjugation.
(ii) $GL(n, \mathbb{R})$ acting on $\mathbb{R}^n$ or $\mathbb{RP}^{n-1}$.
(iii) $O(n)$ (or subgroups of $O(n)$) acting on S^{n-1}.

Definition 6.23: A smooth action of G on a vector space V by linear maps is a smooth representation of G.
This is the same thing as a Lie group homomorphism $\rho: G \to GL(V)$.

Example 6.24: The adjoint representation is the action of G on $\mathfrak{g}$ by conjugation:

$$\mathrm{Ad}_g(\xi) := (C_g)_* \xi$$

The dual representation is the coadjoint.

All actions and representations are smooth from now on.

Definition 6.25: The infinitesimal action of $\xi \in \mathfrak{g}$ on $x \in X$ is

$$\xi \cdot x := D_{(e,x)}\sigma(\xi, 0) = (\exp(t\xi)x)^{\cdot}(0) \in T_x X$$

Example 6.26: The infinitesimal adjoint action of ξ on η is $(\mathrm{Ad}_{\exp(t\xi)}\eta)^{\cdot}(0) = [\xi, \eta]$.

6.4 Quotients and Homogeneous spaces

If a Lie group G acts on a manifold X, then we have a quotient space X/G and a continuous projection $X \to X/G$. Sometimes this quotient is nice e.g. $\mathbb{R}^n \setminus \{0\} / \mathbb{R}^* \cong \mathbb{R}P^{n-1}$, but sometimes its horrible!

e.g. $\mathbb{R}^n / GL(n, \mathbb{R})$ = two points with a non-hausdorff topology

Theorem 6.27 (Lee Theorem 21.10)
If the G action is free and proper, then X/G is a topological manifold of dimension $\dim X - \dim G$, and it has a unique smooth structure that makes $\pi : X \to X/G$ a submersion.

Definition 6.28: The action is proper if the map $G \times X \to X \times X$; $(g, x) \mapsto (x, gx)$ is proper (preimages of compact sets is compact). This is equivalent (Lee prop 21.5) to the following:

if (g_i) and (x_i) are sequences in G and X such that (x_i) and $(g_i x_i)$ converge, then (g_i) has a convergent subsequence.

Definition 6.29: A homogeneous space for G is a manifold X carrying a transitive G-action. A principal homogeneous space is a manifold with a transitive free action, sometimes also called a G-torsor.

If X is a G-torsor, then for any $x \in X$, the orbit map $G \to X$, $g \mapsto gx$ is a diffeomorphism. So X looks like a copy of G but with no distinguished identity element.

Examples 6.30: (i) S^{n-1} is a homogeneous space for $SO(n)$. In fact, its $SO(n)/SO(n-1)$.

(ii) If H is an embedded Lie subgroup of G, then the right / left translation action of H on G is proper, and is obviously free. So G/H is naturally a smooth manifold. The left-translation action of G descends to G/H, making G/H into a homogeneous space. (In fact, every homogeneous space arises in this way)

(iii) The space $F(V)$ of ordered bases in V carries a left action of $GL(V)$, making $F(V)$ into a $GL(V)$-torsor. There is also a right action of $GL(n, \mathbb{R})$, where $n = \dim(V)$, given by: if $e_1, \dots, e_n$ is a basis for V, and $A \in GL(n, \mathbb{R})$, then $(e_1, \dots, e_n)A =: (f_1, \dots, f_n)$ defines a new basis $f_1, \dots, f_n$.

This action is also free and transitive. So $F(V)$ is a $GL(n, \mathbb{R})$ torsor acting on the right.

Recall: action of G on X is free if $\forall x \in X$ if $gx = hx$ then $g = h$.

7 PRINCIPAL BUNDLES AND CONNECTIONS

7.1 Connections by Hand

Fix a vector bundle $\pi : E \to B$ covered by trivialisations Φ_α in the usual way, E has rank k. Given a section s, under Φ_α it becomes an $\mathbb{R}^k$ - valued function v_α. The naive derivative is dv_α, an $\mathbb{R}^k$ - valued 1 - form. Under a different trivialisation Φ_β, v_α becomes $v_\beta = g_{\beta\alpha} v_\alpha$. Let's take the naive derivative of this and then pass the result back to the Φ_α trivialisation:

$$\begin{aligned} g_{\beta\alpha}^{-1}\, dv_\beta &= g_{\beta\alpha}^{-1}\, d(g_{\beta\alpha} v_\alpha) \\ &= g_{\beta\alpha}^{-1} g_{\beta\alpha}\, dv_\alpha + g_{\beta\alpha}^{-1}\, d(g_{\beta\alpha}) v_\alpha \\ &= dv_\alpha + \underbrace{g_{\beta\alpha}^{-1} d(g_{\beta\alpha})\, v_\alpha}_{\text{not necessarily } 0.} \end{aligned}$$

So the result is trivialisation - dependent via the action of the $\mathfrak{gl}(k,\mathbb{R})$ - valued 1 - form on v_α.

> **Elaborate:** let $s : B \to E$ be a section, which locally we can think about as $s : U_\alpha \to E|_{U_\alpha}$. We have a trivialisation $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$. Then
>
> $$\Phi_\alpha \circ s : U_\alpha \to U_\alpha \times \mathbb{R}^k ;$$
>
> $$\begin{tikzcd} p \arrow[r, mapsto, "s"'] & \xi \in E_p \arrow[r, mapsto, "\Phi_\alpha"'] & (p, v_\xi^\alpha) \end{tikzcd} \quad \text{where} \quad v_\xi^\alpha \in \mathbb{R}^k$$
>
> We can then define a function $v_\alpha : B \to \mathbb{R}^k$; $v_\alpha(p) = v_\xi^\alpha$, which is obviously dependent on choice of s.
>
> Notice that for a different trivialisation $p \in U_\beta$, if $\Phi_\beta \circ s : p \mapsto (p, v_\xi^\beta)$, then since $\Phi_\beta \circ \Phi_\alpha^{-1} : (p, \xi) \mapsto (p, g_{\beta\alpha}(p)\xi)$,
>
> $$(p, v_\xi^\beta) = \Phi_\beta \circ s(p) = \Phi_\beta \circ \Phi_\alpha^{-1} \circ \Phi_\alpha \circ s(p) = \Phi_\beta \circ \Phi_\alpha^{-1}(p, v_\xi^\alpha) = (p, g_{\beta\alpha}(p) v_\xi^\alpha)$$
>
> $$\Rightarrow \quad v_\xi^\beta = g_{\beta\alpha}(p) v_\xi^\alpha . \quad \Leftrightarrow \quad v_\beta = g_{\beta\alpha} v_\alpha$$
>
> In some sense this is a canonical way to turn a section into something we can take the derivative of. The (naïve) way to do this would be to just take dv_α (we know $v_\alpha : M \to \mathbb{R}^k$, and we know how to take d of functions like this).
>
> For this derivative to be trivialisation independent, we really want that dv_β and dv_α are related by the transition functions: $dv_\beta = g_{\beta\alpha} dv_\alpha$. But the above says that this is not always the case. So taking the derivative like this is not well defined.

↱ really $\mathcal{A}$ is the collection of these 1-forms that behave on overlaps.

Definition 7.1 (preliminary version) A **connection $\mathcal{A}$** on E is a $\mathfrak{gl}(k, \mathbb{R})$-valued 1-form A_α on each trivialisation patch $U_\alpha \subset B$ such that on overlaps

$$A_\alpha = g_{\beta\alpha}^{-1} dg_{\beta\alpha} + g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha}$$

The covariant derivative of a section s with respect to $\mathcal{A}$ is the E-valued 1-form $d^{\mathcal{A}} s$ defined locally under Φ_α by $dv_\alpha + A_\alpha v_\alpha$.

> Consider the local trivialisation $\Phi_\alpha : \pi^{-1}(U_\alpha)(\subset E) \to U_\alpha \times \mathbb{R}^k$ (really every $\xi \in E$ can be thought of as a vector at a point). So how is $d^{\mathcal{A}} s$ an E-valued one form? It's defined under Φ_α as $dv_\alpha + A_\alpha v_\alpha$, which pulls back to give a one form $(\Phi_\alpha)^*(dv_\alpha + A_\alpha v_\alpha)$ ($\mathbb{R}^k$-valued 1-form to an E-valued 1-form).
>
> Let's analyse $dv_\alpha + A_\alpha v_\alpha$. Now, v_α is an $\mathbb{R}^k$-valued function, i.e. $v_\alpha : B \to \mathbb{R}^k$. So dv_α is an $\mathbb{R}^k$-valued 1-form (coefficients are in $\mathbb{R}^k$, or rather maps $B \to \mathbb{R}^k$). What we really need to convince ourselves of however is that $A_\alpha v_\alpha$ is an $\mathbb{R}^k$-valued 1-form.
>
> Now A_α is a $\mathfrak{gl}(k, \mathbb{R})$-valued 1-form, and so locally it has coefficients given by maps $B \to \mathfrak{gl}(k, \mathbb{R})$ (so matrices dependent smoothly on $p \in B$).
>
> Say $A_\alpha = \sum M_i \, dx^i$. Then $A_\alpha(v_\alpha) = \sum M_i(v_\alpha) \, dx^i$, where we mean at $p \in B$, $= \sum M_i|_p(v_\alpha(p)) \, dx^i$. Hence $A_\alpha v_\alpha$ is an $\mathbb{R}^k$-valued 1-form
>
> So this all makes sense... pretty much. The condition above requires that A_α behaves nicely (agrees) on overlaps. But of course, this agreement is under the gluing maps. Passing to the trivialisation, this says that $dv_\beta + A_\beta v_\beta = g_{\beta\alpha}(dv_\alpha + A_\alpha v_\alpha)$. The next part says that this condition is exactly what we need:

This is consistent on overlaps:

$$\begin{aligned} g_{\beta\alpha}^{-1}(dv_\beta + A_\beta v_\beta) &= g_{\beta\alpha}^{-1} d(g_{\beta\alpha} v_\alpha) + g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} v_\alpha \\ &= d(v_\alpha) + \underbrace{g_{\beta\alpha}^{-1}(dg_{\beta\alpha}) v_\alpha + g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} v_\alpha}_{A_\alpha(v_\alpha)} \end{aligned}$$

We say s is **horizontal / covariantly constant** if $d^{\mathcal{A}} s = 0$.

Example 7.2: Suppose E splits as $F \oplus F'$ for some rank-ℓ subbundle F. We can cover E by trivialisations Φ_α in which the splitting becomes the ordinary splitting in $\mathbb{R}^k$: $\mathbb{R}^k = \mathbb{R}^\ell \oplus \mathbb{R}^{k-\ell}$.

Given a connection A on E, we can define a connection on F by taking the top left $\ell \times \ell$ submatrix of each A_α (restricting the 1-forms).

The covariant derivative of a section s of F is given by taking $d^{\mathcal{A}} s$ in E and projecting onto F along F'. In particular, if $\iota : X \hookrightarrow \mathbb{R}^N$ is an embedding, then $E = \iota^* T\mathbb{R}^N$ has a canonical trivialisation Φ_α and hence a canonical connection with $A_\alpha = 0$.

The splitting $E = TX \oplus TX^\perp$ then induces a connection on TX.

Definition 7.3 : The frame bundle $F(E)$ of E is the space of ordered bases in each fibre. I.e

$$F(E) := \bigsqcup_{\alpha} U_\alpha \times F(\mathbb{R}^k) / \sim \qquad \text{where } (b \in U_\alpha,\ v_1, \dots, v_k) \sim (b \in U_\beta,\ g_{\beta\alpha}(b) v_1, \dots, g_{\beta\alpha}(b) v_k)$$

This has a projection $\pi_F : F(E) \to B$. It carries a right $GL(k,\mathbb{R})$-action, making every fibre $\pi_F^{-1}(b)$ into a principal homogeneous space. A section of $F(E)$ over U is a map $f: U \to F(E)$ such that $\pi_F \circ f = id_U$.

> The frame bundle has a natural right action of $GL(k,\mathbb{R})$ which is given by an ordered change of basis, which is free and transitive. Since it acts on the right, it doesn't interfere with the gluing map $\sim$:
>
> $$(v_1, \dots, v_k) \sim (g_{\beta\alpha}(p) v_1, \dots, g_{\beta\alpha}(p) v_k)$$
>
> and $(v_1 M, \dots, v_k M) \sim ((g_{\beta\alpha} v_1) M, \dots, (g_{\beta\alpha}(p) v_k) M)$ for a change of basis matrix M

Note : sections of $F(E)$ over U correspond to trivialisations of E over U.

> A section of $F(E)$ is an assignment of bases in each fibre in some smooth way that agrees on the overlaps. So really locally this is a way of writing down maps $\to \mathbb{R}^k$ on each chart U.
>
> So over U, for each $\xi \in E$, we can write the point as some vector in terms of our chosen basis, which gives a $v_\xi \in \mathbb{R}^k$. Hence we can identify (locally), every $\xi \in E \longleftrightarrow v_\xi \in \mathbb{R}^k$, in a smooth way.

Let $f_\alpha : U_\alpha \to F(E)$ be the section of $F(E)$ corresponding to the trivialisation Φ_α of E. (Φ_α, U_α)

We get for each α a diffeomorphism $\Phi_\alpha^F : \pi_F^{-1}(U_\alpha) \to U_\alpha \times GL(k,\mathbb{R})$;

$$f_\alpha(b) \underbrace{g}_{\text{apply a change of basis.}} \longmapsto (\underset{\in U_\alpha}{b}, \underset{\text{matrix}}{g})$$

$g \in GL(k,\mathbb{R})$ is invertible, acting on the right $\Rightarrow$ diffeomorphism is well defined.

Take a connection $\mathcal{A}$ on E. For each α we can build a $\mathfrak{gl}(k,\mathbb{R})$-valued 1-form on $U_\alpha \times GL(k,\mathbb{R})$ as follows:

$$(v \in T_b U_\alpha,\ g \cdot \xi \in T_g GL(k,\mathbb{R})) \longmapsto \underset{\substack{\text{conjugate by } g^{-1} \\ (\text{i.e. } g^{-1} A_\alpha(v) g)}}{Ad_{g^{-1}}} A_\alpha(v) + \xi$$

Pulling back by Φ_α^F gives a $\mathfrak{gl}(k,\mathbb{R})$-valued 1-form on $\pi_F^{-1}(U_\alpha)$.

> Let's think about how we define 1-forms. If we want it to be defined on $U_\alpha \times GL(k,\mathbb{R})$, we want a map that takes vector fields to maps $U_\alpha \to \mathfrak{gl}(k,\mathbb{R})$ (Essentially coefficients of 1-form are maps $U_\alpha \to \mathfrak{gl}(k,\mathbb{R})$)
>
> Equivalently, we can define how the 1-form (when evaluated at a point) acts on tangent vectors at that point, under the assumption that its dependence is smooth.
>
> Let $g \cdot \xi \in T_g GL(k,\mathbb{R})$. For $g \in GL(k,\mathbb{R})$, g is a matrix, and we can identify $GL(k,\mathbb{R}) \sim \mathbb{R}^{k^2}$ to see that $g \cdot \xi \in T_g GL(k,\mathbb{R})$ is a matrix too (notice that we don't have to write $g \cdot \xi$, we could just write ξ but for our purposes we're making use of it).
>
> $(v, g \cdot \xi) \longmapsto Ad_{g^{-1}} A_\alpha(v) + \xi$ gives us something in $\mathfrak{gl}(k,\mathbb{R})$

Prop 7.4: These local constructions agree on overlaps, and define a $\mathfrak{gl}(k,\mathbb{R})$ - valued 1-form $\mathcal{A}$ on $F(E)$ satisfying

- $\mathcal{A}_{\underset{p=(\text{point, basis})}{p}}(p\cdot\xi) = \xi \quad \forall\, p \in F(E), \quad \xi \in \underset{\text{matrix}}{\mathfrak{gl}(k,\mathbb{R})}.$
- $R_g^*\mathcal{A} = \mathrm{Ad}_{g^{-1}}\mathcal{A}$ for all $g \in GL(k,\mathbb{R})$.

Conversely, any $\mathfrak{gl}(k,\mathbb{R})$ - valued 1-form $\mathcal{A}$ on $F(E)$ satisfying these two conditions defines a connection on E, according to Dfn 7.1, via $A_\alpha = f_\alpha^*\mathcal{A}$.

Definition 7.5: A connection on E is a $\mathfrak{gl}(k,\mathbb{R})$ - valued 1-form on $F(E)$ satisfying these two conditions.

7.2. Principal Bundles

Fix a Lie group G.

Definition 7.6: A (principal) G - bundle over a manifold B is a manifold P equipped with

- A smooth surjection $\pi : P \to B$
- A collection of open sets U_α covering B, and for each α a diffeomorphism

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G.$$

so each fibre is a copy of G.

want this to commute:

$$\begin{tikzcd}
\pi^{-1}U_\alpha \arrow[r, "\Phi_\alpha"] \arrow[d, "\pi"'] & U_\alpha \times G \arrow[dl, "pr_1"] \\
U_\alpha &
\end{tikzcd}$$

Such that :
1) $pr_1 \circ \Phi_\alpha = \pi$ (restricted to $\pi^{-1}(U_\alpha)$).
2) $\Phi_\beta \circ \Phi_\alpha^{-1}(b,g) = (b, g_{\beta\alpha}(b)g)$ for some smooth maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$.

P is total space, B is base, Φ_α are trivialisations, $g_{\beta\alpha}$ are transition functions, etc.
Lots of concepts carry over from vector bundles, e.g. pullbacks, sections, construction by gluing.

Each trivialisation Φ_α gives a section $b \mapsto \Phi_\alpha^{-1}(b,e)$, over U_α. Conversely, a section s over U defines a trivialisation over U via $\Phi^{-1}(b,g) = s(b)g$

Here we're using right G - action on P

> A trivialisation $\Phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times G$. If s is a section $B \to P$, then for $(b,g) \in U_\alpha \times G$, we can get an element of $\pi^{-1}(U_\alpha)$ by letting s act on b ($s(b) \in P$), and then having $g \in G$ act on it. We need to think a little about why this is a diffeomorphism. First, is this even well-defined? Well yeah. A section s is such that $\pi \circ s(b) = b$, and so for some $s(b) \in P$, you can always recover the original point b by composing with π. Now letting g act on $s(b)$, by Example 6.30 iii), this action is free and transitive (I think ?) which means that $s(b)g = s(b)h \Leftrightarrow g = h$, so that we can recover g by this uniqueness. Of course, all of this is smooth, so Φ is a diffeomorphism. need to check trivialisation conditions

If $P \to B$ is a principal G bundle, then P has a right action, defined in trivialisations:
i.e. if $\Phi_\alpha(p) = (b,g)$, then $p\cdot h = (b, gh)$. This gives a correspondence between sections of P and trivialisations

$$\left\{ \Phi \longrightarrow s \quad \text{defined by} \quad s(b) = \Phi^{-1}(b,g) \right\}$$
$$\downarrow\uparrow$$
$$\left\{ s \longrightarrow \Phi \quad \text{defined by} \quad \Phi^{-1}(b,g) = s(b)g \right\}$$

Example 7.7 (i) if E is a rank-k vector bundle over B, then $F(E)$ is a principal $GL(k,\mathbb{R})$-bundle (action corresponds to change in basis)

(ii) $B\times G \to B$ is the trivial G-bundle

(iii) A G-bundle over a point is just a G-torsor.

Warning! A rank-k vector bundle is not the same as a principal $\mathbb{R}^k$-bundle.

> For a vector bundle, the trivialisations are glued along intersections via isomorphisms of vector spaces (elements of $GL(k,\mathbb{R})$). But for a principal $\mathbb{R}^k$-bundle, the gluing is done by elements of $\mathbb{R}^k$ (translations).
>
> Remember: transition functions of a vector bundle: $g_{\beta\alpha}: U_\alpha\cap U_\beta \to GL(k,\mathbb{R})$
> transition functions of a $\mathbb{R}^k$-bundle : $g_{\beta\alpha}: U_\alpha\cap U_\beta \to G=\mathbb{R}^k$. } not the same!

↳ identify $p\sim pg$ $\forall g\in G$.

The right G-action on a G-bundle P is free and proper, and P/G is B.
Conversely, if P is a manifold, carrying a free and proper right G-action, then the quotient map $\pi: P\to P/G$ gives a principal G-bundle (π is a submersion, so has local sections, and they induce trivialisations via the right G-action).

Example 7.8: Recall the Hopf map $H: S^{2n+1}\to \mathbb{C}P^n$. The sphere $S^{2n+1}\subset\mathbb{C}^{n+1}$ carries a free $U(1)$ action which is also proper since $U(1)$ is compact. The action is by scalar multiplication and the quotient map is H. Hence H is a principal $U(1)$ bundle.

Definition 7.9: If $P\to G$ is a G-bundle and $\rho: G\to GL(V)$ is a representation of G, then the associated vector bundle is

$$P\times_G V = {P\times V}/{(pg,v)\sim(p,\rho(g)v)}$$

gives a v.b. over B

If P is trivialised over U_α with transition functions $g_{\beta\alpha}$, then $P\times_G V$ is trivialised over the same U_α with transition functions $\rho(g_{\beta\alpha})$.

> Note that $g_{\beta\alpha}: U_\alpha\cap U_\beta\to G$, so $\rho\circ g_{\beta\alpha}: U_\alpha\cap U_\beta \to GL(V)$, and this is enough to define a vector bundle.

Example 7.10

i) if $P=F(E)$, and $\rho: GL(k,\mathbb{R})\to GL(k,\mathbb{R})$ is the identity, then the associated v.b. is E itself.

> Then $G=GL(k,\mathbb{R})$, which acts as a change of basis on the right. We can think of a point in $F(E)$ as $(b, v^1,\dots,v^n)$, then $pg=(b,v^1g,\dots,v^ng)$. For a vector $w\in\mathbb{R}^k$ then, we get the identification $(b,v^1g,\dots,v^ng,w)\sim(b,v^1,\dots,v^n,gw)$, which encodes the exact same data as E.

ii) if $P=F(E)$, and ρ is the dual representation (transpose inverse), then the associated v.b is $E^\vee$.
Similarly we can get tensor powers of $E, E^\vee$.

iii) if $\rho: G\to GL(\mathfrak{g})$ is the adjoint representation $(\rho(g)\xi = g\xi g^{-1})$, then the associated v.b is called the adjoint bundle $\mathrm{ad}P$.
If $P=F(E)$, then $\mathrm{ad}(P)=\mathrm{End}(E)=E^\vee\otimes E$

7.3 Connections

Let $\pi : P \to B$ be a G-bundle

Definition 7.11: A connection on P is a $\mathfrak{g}$-valued 1-form $\mathcal{A}$ on P satisfying

- $\mathcal{A}_p(\underset{T_pP}{\underset{\cap}{p\cdot\xi}}) = \xi$
- $\boxed{R_g^*}\,\mathcal{A} = \mathrm{Ad}_{g^{-1}}\mathcal{A}$

$R_g : P \to P;$
$p \mapsto pg$

If Φ_α is a trivialisation of P corresponding to a section s_α, then

$$A_\alpha := s_\alpha^* \mathcal{A}$$

is called the local connection 1-form.

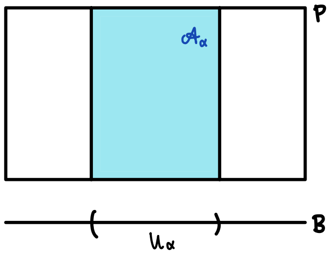
> $\mathcal{A}$ is a 1-form on P, and $s_\alpha : B \to P$, so that $s_\alpha^*\mathcal{A}$ is a 1-form on B.
>
> NB: Recall that $p\cdot\xi$ for $p \in P$ and $\xi \in \mathfrak{g}$ means the infinitesimal action of $\xi \in \mathfrak{g}$ on P at p (dfn 6.25)
> This is the most natural way to get a tangent vector $\in T_pP$ from one $\xi \in T_eG = \mathfrak{g}$.

Lemma 7.12: On overlaps, $A_\alpha = g_{\beta\alpha}^{-1}\, d g_{\beta\alpha} + \mathrm{Ad}_{g_{\beta\alpha}^{-1}} A_\beta$

let $p \in U_\alpha \cap U_\beta$. Then A

Proposition 7.13: Every principal bundle (and hence every vector bundle by considering frame bundles) admits a connection.

proof: We can cover P by trivialisations Φ_α over U_α, and define a connection $\mathcal{A}_\alpha$ on $\pi^{-1}(U_\alpha)$ by taking $A_\alpha = 0$



Basically build up $\mathcal{A}$ using local connections that satisfy overlap conditions

Let $\{\rho_\alpha\}$ be a partition of unity sub to this cover. Then $\mathcal{A} := \sum_\alpha (\rho_\alpha \circ \pi)\mathcal{A}_\alpha$ defines a connection on P:

- For $p \in P$, $\xi \in \mathfrak{g}$ we have $\mathcal{A}(p\cdot\xi) = \sum_\alpha \underbrace{\rho_\alpha \circ \pi(p)}_{=0 \text{ outside } U_\alpha} \underbrace{\mathcal{A}_\alpha(p\cdot\xi)}_{=\xi \text{ on } U_\alpha} = \sum_\alpha \rho_\alpha \circ \pi(p)\, \xi = \xi.$
- For $g \in G$, we have

$$\begin{aligned} R_g^*\mathcal{A} &= \sum_\alpha \rho_\alpha \circ \pi\, R_g^*\mathcal{A}_\alpha \\ &= \sum_\alpha \rho_\alpha \circ \pi\, \mathrm{Ad}_{g^{-1}}\mathcal{A}_\alpha \\ &= \mathrm{Ad}_{g^{-1}} \sum_\alpha \rho_\alpha \circ \pi\, \mathcal{A}_\alpha \\ &= \mathrm{Ad}_{g^{-1}}\mathcal{A}. \end{aligned}$$

☐

Proposition 7.14: The space of all connections on P is a torsor for the space of $\mathrm{ad}P$-valued 1-forms on B.

= Homogeneous principal space := space with a transitive G-action

pf: Fix a reference connection $\mathcal{A}^\circ$ on P. Now let $\mathcal{A}$ be any other connection. Consider the $\mathfrak{g}$-valued 1-forms $A_\alpha - A_\alpha^\circ$ on $U_\alpha \subset B$. On overlaps, we have

$$A_\alpha - A_\alpha^\circ = \mathrm{Ad}_{g_{\beta\alpha}}(A_\beta - A_\beta^\circ)$$

So they glue together to give an $\mathrm{ad}P$-valued 1-form. Conversely, if D is an $\mathrm{ad}P$-valued 1-form, then the $\mathfrak{g}$-valued 1-forms $A_\alpha^\circ + D_\alpha$ define a connection $\mathcal{A}$. These two constructions are inverse. $\square$

Definition 7.15: For $p \in P$, the vertical subspace at p is $T_p^{\overset{\swarrow\, v \text{ for vert, not dual!}}{v}} P = \ker D_p\pi = TP_{\pi(p)} = p\cdot\mathfrak{g}$

A horizontal subspace is any complementary subspace.

A horizontal distribution is a distribution H on P which is a horizontal subspace at every point.

Given a connection $\mathcal{A}$ on P, $H := \ker\mathcal{A}$ is a horizontal distribution:

rank-nullity $\Rightarrow$ $\dim(\ker\mathcal{A}) = \dim(P) - \dim(\mathfrak{g}) = \dim(P) - \dim(T^vP)$

Also $\ker\mathcal{A} \cap T^vP = 0$ since if $p\cdot\xi$ is in $\ker\mathcal{A}$, then

$$\mathcal{A}(p\cdot\xi) = 0 \quad \text{but} \quad \mathcal{A}(p\cdot\xi) = \xi$$

Because $\mathcal{A}$ is right equivariant, H is right-invariant, i.e. $(R_g)_* H = H$.
Conversely, given a right invariant horizontal distribution H, $\exists$ a unique connection $\mathcal{A}$ with $\ker\mathcal{A} = H$.

Any vector can be decomposed uniquely as $p\cdot\overset{\overset{\mathfrak{g}}{\cup\!\shortmid}}{\xi} + \overset{\overset{H}{\cup\!\shortmid}}{h}$. Then define $\mathcal{A}(v) = \xi$. A section s of P is horizontal iff it's tangent to the horizontal distribution, i.e. $s^*\mathcal{A} = 0$.

Example 7.16

(i) Consider the projection $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ $(x,y,z) \longmapsto (x,y)$ as a (trivial) principle $\mathbb{R}$-bundle. The distributions $\langle \partial_x, \partial_y\rangle$ and $\langle \partial_x + y\partial_z, \partial_y\rangle$ are horizontal (don't contain ∂_z), and are $\mathbb{R}$-invariant (invariant under translation in z direction). So they each define a connection on the bundle.

Case 1: $\mathcal{A} = \ker\langle\partial_x, \partial_y\rangle = dz$ $\quad (A = 0)$

case 2: $\mathcal{A} = \ker\langle\partial_x + y\partial_z, \partial_y\rangle = dz - ydx$ $\quad (A = -ydx)$

(ii) Recall the Hopf bundle $H: \underset{\underset{\mathbb{C}^{n+1}}{\cap}}{S^{2n+1}} \longrightarrow \mathbb{C}P^n$

View T_pS^{2n+1} as a subspace of $\mathbb{C}^{n+1}$. Consider $T_pS^{2n+1} \cap i\cdot T_pS^{2n+1}$. This defines a $U(1)$-invariant horizontal distribution, hence a connection.

Recall a section of E is horizontal iff covariantly constant. Can check that a connection on E induces a horizontal distribution on E s.t a section is horizontal in the old sense (covariantly constant) iff its tangent to this distribution. Recall also that a section of $F(E)$ is a k-tuple of sections $s_1, \dots, s_k$ of E. Then f is horizontal iff the s_i are horizontal.

Using the horizontal distribution, we can define parallel transport on $P \to B$ or $E \to B$ as on Example sheet 3 Q7.

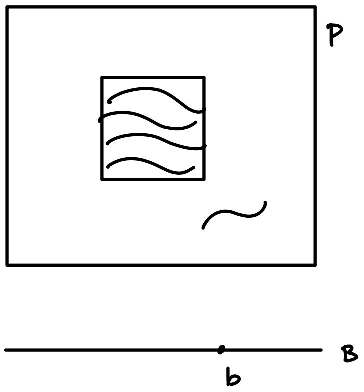
7.4. Curvature

Fix a principal G-bundle $P \to B$ with a connection $\mathcal{A}$.

Definition 7.17: $\mathcal{A}$ is flat iff the horizontal distribution is integrable (arises from a foliation).

Proposition 7.18: the following are equivalent

(i) $\mathcal{A}$ is flat
(ii) P is foliated by local horizontal sections
(iii) P has a horizontal section locally over each point in B.
(iv) P can be covered by trivialisations Φ_α such that all A_α are 0.



proof: (i) $\Leftrightarrow$ (ii)

(ii) just spells out what it means for the horizontal distribution to arise from a foliation

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (ii) Given $p \in P$, by (iii) $\exists$ horizontal section s over $U \ni \pi(p)$. Then the right translates of s foliate P over U.

(iii) $\Leftrightarrow$ (iv): Given a trivialisation Φ_α, the corresponding section s_α is horizontal iff $\underset{A_\alpha}{\underset{\shortparallel}{s_\alpha^* \mathcal{A}}} = 0$

$\square$

Curvature is the obstruction to flatness.

$d^{\alpha} s = 0 \Leftrightarrow s_\alpha^{+} \mathcal{A} = 0$

Definition: The curvature $\overset{\text{curly } F}{\mathcal{F}}$ of $\mathcal{A}$ is the $\mathfrak{g}$-valued 2-form

$dv_\alpha + A_\alpha v_\alpha$

$$d\mathcal{A} + \frac{1}{2}[\mathcal{A} \wedge \mathcal{A}]$$

$\Leftrightarrow \quad dv_\alpha = 0$

Notation: For $\mathfrak{g}$-valued p, q forms $\sigma = \sum_i \xi_i \otimes \sigma_i$, $\tau = \sum_i \eta_j \otimes \tau_j$, we write $[\sigma \wedge \tau]$ for

$$[\sigma \wedge \tau] = \sum_{i,j} [\xi_i, \eta_i] \otimes (\sigma_i \wedge \tau_j)$$

Warning: $[\sigma \wedge \tau] = (-1)^{pq+1}[\tau \wedge \sigma]$. equiv: $[\sigma \wedge \tau](X_1, \dots, X_n) = \sum_{\sigma \in S_{p+q}} (-1)^{\operatorname{sgn}(\sigma)} [\sigma(X_{\sigma(1)}, \dots, X_{\sigma(p)}), \eta(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)})]$

> If $\mathcal{A}$ is a $\mathfrak{g}$-valued 1-form, then
>
> $$[\mathcal{A} \wedge \mathcal{A}](X_1, X_2) = [A(X_1), A(X_2)] - [A(X_2), A(X_1)]$$
> $$= 2[A(X_1), A(X_2)]$$

Theorem 7.20 : $\mathcal{A}$ is flat $\Leftrightarrow$ $\mathcal{F} = 0$.

proof: we claim $\mathcal{F}(v,w) = 0$ if (WLOG) v is vertical. Then by Frobenius, $\mathcal{A}$ is flat iff $d\mathcal{A} \in I(\ker \mathcal{A})$ $\Leftrightarrow$ $d\mathcal{A}(v,w) = 0$ $\forall$ horizontal v,w.

> Idea: let $X = v_1 + v_2$ be some tangent vector (or vector field if you like) with v_1 vertical and v_2 horizontal. We want to show that the curvature map $\mathcal{F}$ is 0 iff $\mathcal{A}$ is flat (dist$^{\underline{n}}$ arises from a foliation. We can check some cases:
>
> $\mathcal{F}(v,w)$: where v vertical → dont care about what w is
> $\mathcal{F}(v,w)$: where v and w are both horizontal
> } these cover all cases by linearity of $\mathcal{F}$.
>
> And so if we know that $\mathcal{F}(v,w) = 0$ $\forall$ v vertical, then to see that $\mathcal{F} = 0$, we just have to check the condition just for v,w horizontal. This gives us our equivalence.
>
> Suppose we want $\mathcal{F}(v,w) = 0$. Then says $d\mathcal{A}(v,w) + \frac{1}{2}[\mathcal{A}\wedge\mathcal{A}](v,w) = 0$. Now,
>
> $\frac{1}{2}[\mathcal{A}\wedge\mathcal{A}](v,w) = [\mathcal{A}(v), \mathcal{A}(w)]$, but when w is horizontal, $\Rightarrow \mathcal{A}(w) = 0$ ($w \in \ker \mathcal{A}$), and hence this is equivalent to $d\mathcal{A}(v,w) = 0$.

$\Leftrightarrow$ $\mathcal{F}(v,w) = 0$ $\forall$ horizontal v,w
Since $[\mathcal{A}\wedge\mathcal{A}]$ vanishes on horizontal vectors)

$\Leftrightarrow$ (by claim) $\mathcal{F}(v,w) = 0$ $\forall v,w$.

It remains to prove the claim, so let v be the vertical vector field $v(p) = p\cdot\xi$ (ξ fixed).
We want to prove $\iota_v\mathcal{F} = 0$. We have

$$\iota_v\mathcal{F} = \iota_v d\mathcal{A} + [\mathcal{A}(v), \mathcal{A}]$$

$$= \iota_v d\mathcal{A} + [\xi, \mathcal{A}]$$

So its left to show $[\xi, \mathcal{A}] = -\iota_v d\mathcal{A}$.

We have $[\xi, \mathcal{A}] = \frac{d}{dt}\Big|_{t=0} Ad_{\exp(t\xi)}\mathcal{A}$. $\qquad R_g^*\mathcal{A} = Ad_g\text{-}\mathcal{A}$.

$$= \frac{d}{dt}\Big|_{t=0}\left(R_{\exp(-t\xi)}\right)^*\mathcal{A}.$$

$$= -\mathcal{L}_v\mathcal{A}$$

$$= -\iota_v d\mathcal{A} + \boxed{d\iota_v\mathcal{A}} \qquad = d\xi = 0$$

$$= -\iota_v d\mathcal{A}.$$

$\square$

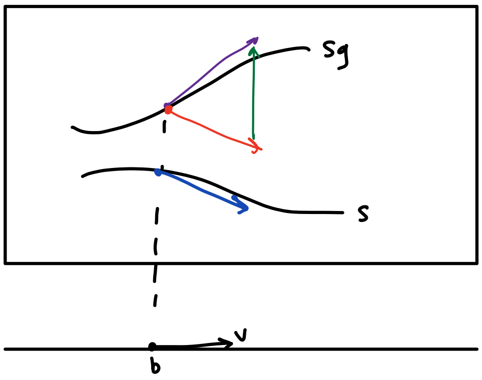
Given a section s_α corresponding to a trivialisation Φ_α, we write F_α for $s_\alpha^* \mathcal{F}$. Then F_α is a $\mathfrak{g}$-valued 2-form on U_α.

Proposition 7.21: These local expressions glue together to give an $\mathrm{ad}P$-valued 2-form on B.

proof: on overlaps we have $s_\beta = s_\alpha g_{\beta\alpha}^{-1}$, and we want to show $F_\beta = \mathrm{Ad}_{g_{\beta\alpha}} F_\alpha$.

Let $s = s_\alpha$, $g = g_{\beta\alpha}^{-1}$. For any vector $v \in T_b(U_\alpha \cap U_\beta)$,

- $\underline{(sg)_*} v - \underline{(R_g)_* (\underline{s_* v})}$ is vertical.



Easier to work at a fixed $b \in U_\alpha \cap U_\beta$ and say

$$R_{g(b)}^* \mathcal{F} = \mathrm{Ad}_{g(b)^{-1}} \mathcal{F}$$

Since $\mathcal{F}$ annihilates vertical vectors, we get

$$\begin{aligned} (sg)^* \mathcal{F} &= s^* R_g^* \mathcal{F} \\ \| \quad & \quad\ \| \\ F_\beta \quad & \quad s^* \mathrm{Ad}_{g^{-1}} \mathcal{F} \\ & \quad\ \| \\ & \quad \mathrm{Ad}_{g^{-1}} F_\alpha \end{aligned}$$

since.
$R_g^* \mathcal{A} = \mathrm{Ad}_{g^{-1}} \mathcal{A}$

$\square$

Example 7.22: For our two connections on our trivial $\mathbb{R}$-bundle $\pi: \mathbb{R}^3 \to \mathbb{R}^2$, we have

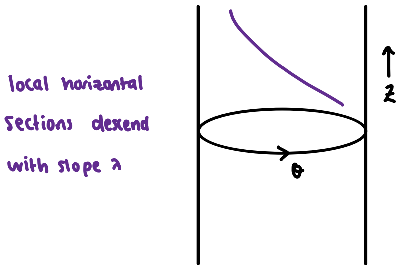
$F = 0 \quad (A = 0)$
$F = dx \wedge dy \quad (A = -y\,dx)$

Notation

$\frac{1}{2}[A_\alpha \wedge A_\alpha](v,w)$ wedge bracket
$= \frac{1}{2}[A_\alpha(v), A_\alpha(w)] - \frac{1}{2}[A_\alpha(w), A_\alpha(v)]$ commutator

$= [A_\alpha(v), A_\alpha(w)]$ commutator
$= A_\alpha(v) A_\alpha(w) - A_\alpha(w) A_\alpha(v)$

Warning! Even if $\mathcal{F} = 0$ everywhere, global horizontal sections may not exist. E.g. take the trivial principal $\mathbb{R}$-bundle over S^1 with $A = \lambda d\theta$, and fibre coordinate z (so $\mathcal{A} = dz + \lambda d\theta$)



So if $\lambda \neq 0$, then $\nexists$ global horizontal section

7.5 Algebraic Structures

Given a connection $\mathcal{A}$ on a G-bundle $P \to B$ and a representation $\rho : G \to GL(V)$, there's an induced connection on the associated vector bundle $E = P \times_G V$

Its defined by local connection 1-forms $D_e\rho(A_\alpha)$

Example 7.23: If P is the frame bundle of some vector bundle F, then a connection on P induces connections on $F^\vee$, $F \otimes F^\vee$, etc.

Can also extend the covariant derivative $d^{\mathcal{A}}$ to an exterior covariant derivative using the Leibniz rule: on E valued p-form σ can locally be written as a sum of expressions $s \otimes \alpha$ where s is a section of E and α is a p-form. Then define $d^{\mathcal{A}}(s \otimes \alpha)$ to be $(d^{\mathcal{A}} s) \wedge \alpha + s \otimes d\alpha$

Proposition 7.24 ((second) Bianchi identity) $d^{\mathcal{A}} F = 0$

(Here F is an $\operatorname{ad}P$-valued 2-form on B, and $d^{\mathcal{A}}$ is the exterior covariant derivative)

proof: Locally in a trivialisation, we write F as F_α, a $\mathfrak{g}$-valued 2-form. Then locally

$$d^{\mathcal{A}} F = dF_\alpha + (\underset{\nearrow \text{ equiv. to } D_e\rho \text{ in this context}}{\operatorname{ad} A_\alpha}) \wedge F_\alpha$$

$$= dF_\alpha + [A_\alpha \wedge F_\alpha]$$

$$= \underbrace{d^2 A_\alpha}_{0 \text{ since } d^2 = 0} + \underbrace{\tfrac{1}{2} d[A_\alpha \wedge A_\alpha] + [A_\alpha \wedge dA_\alpha]} + \underbrace{[A_\alpha \wedge \tfrac{1}{2}[A_\alpha \wedge A_\alpha]]}_{= 0 \text{ by Jacobi identity}}$$

$\tfrac{1}{2} d[A_\alpha \wedge A_\alpha] : \tfrac{1}{2}[dA_\alpha \wedge A_\alpha] - \tfrac{1}{2}[A_\alpha \wedge dA_\alpha]$

$= -[A_\alpha \wedge dA_\alpha]$

so terms cancel out.

Warning! $(d^{\mathcal{A}})^2 \neq 0$ in general. In fact, $(d^{\mathcal{A}})^2 \underset{\substack{\nearrow \\ E\text{-valued} \\ p\text{-form}}}{\sigma} = \underbrace{D_e\rho(F)}_{\substack{\operatorname{End}(E)\text{-valued} \\ 2\text{-form}}} \wedge \sigma$

8 RIEMANNIAN GEOMETRY

8.1 Metrics

Given a vector bundle $E \to B$, Sections of $(E^\vee)^{\otimes 2}$ Correspond to fibrewise bilinear forms on E.

Definition 8.1: An inner product g on E is a section of $(E^\vee)^{\otimes 2}$ which is fibrewise Symmetric and positive definite (i.e. an inner product on each fibre).

A Riemannian metric on X is an inner product on TX.

Lemma 8.2: Every vector bundle $E \to B$ admits an inner product. Hence every manifold admits a Riemannian metric.

proof: Cover E with trivialisations $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$

On each $\pi^{-1}(U_\alpha)$ there's an inner product g_α corresponding to the standard inner product on $\mathbb{R}^k$. Take a partition of unity $\{\rho_\alpha\}$ and set $g = \sum \rho_\alpha g_\alpha$.

Definition 8.3: A Riemannian manifold (X, g) is a manifold equipped with a Riemannian metric.

Write $g = g_{ab}$. Let g^{ab} be the dual metric, defined by $g^{ab} = g^{ba}$, $g^{ab} g_{bc} = \delta^a{}_c$

Write contraction with g_{ab}, g^{ab} by raising/lowering indices

e.g. $g^{bd} T^a{}_{bc} = T^{ad}{}_c$

Notation : $dx^i dx^j = \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i)$

Definition 8.4: A connection $\mathcal{A}$ on E is compatible with an inner product g if g is covariantly constant wrt the induced connection on $(E^\vee)^{\otimes 2}$

8.2 Connections on TX

Fix a manifold X

Definition 8.5: A connection on X is a connection on TX. We'll think of this as a connection on E, where E is identified with TX via an E-valued 1-form θ.

For $x \in X$, $\theta_x \in E_x \otimes T_x^* X = \mathrm{Hom}(T_x X, E_x)$ Usually the covariant derivative is written ∇, and its connection with a vector v is written ∇_v. In local coordinates, $\theta = \partial_{x_i} \otimes dx^i$

Definition 8.6: The torsion of a connection $\mathcal{A}$ on $E = TX$ is $d^{\mathcal{A}}\theta$, an E-valued 2-form.

(Sheet 4 : $\nabla_v w - \nabla_w v = [v, w] + T(v, w)$)

The connection is called torsion free if $T = 0$.

Proposition 8.7 (First Bianchi identity) $d^{\mathcal{A}} T = F \wedge \Theta$, where F is the $\mathrm{End}(E)$-valued curvature 2-form on X.

proof: both sides are $(d^{\mathcal{A}})^2 \Theta$ $\square$

Theorem 8.8 (Fundamental Theorem of Riemannian Geometry)

Given a Riemannian manifold (X, g), theres a unique torsion free connection on X compatible with g. This is known as the Levi-Civita connection

proof: We'll show that the map

$$\left\{ \begin{matrix} g\text{-compatible} \\ \text{connections} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} E\text{-valued} \\ 2 \text{ forms} \end{matrix} \right\}$$

$$\mathcal{A} \longrightarrow T_{\mathcal{A}}$$

is a bijection.

Let $F_0(E)$ be the orthogonal frame bundle of E - a principal $O(n)$-bundle. Note that E is an associated vector bundle of $F_0(E)$, $E = F_0 E \times_{O(n)} \mathbb{R}^n$ via the representation of $O(n)$. So connections on $F_0(E)$ induce connections on E is compatible with gg iff it arises in this way (Example sheet 4)

Fix a connection $\mathcal{A}_0$ on $F_0(E)$. We get a bijection

$$\{ g\text{-compatible connections on } X \} \longrightarrow \{ \mathrm{ad}\, F_0(E) \text{-valued 1-forms on } X \}$$

$$\mathcal{A} \longrightarrow \Delta := \mathcal{A} - \mathcal{A}_0$$

gothic o

We also have $\mathrm{ad}\, F_0(E) \cong \mathfrak{o}(E) = \{ \text{skew-adjoint endomorphisms of } E \}$

So its left to show that

$$\{ \mathfrak{o}(E)\text{-valued 1-forms on } X \} \to \{ E\text{-valued 2-forms} \} \quad \text{is a bijection}$$

$$\Delta \longmapsto T_{\mathcal{A}_0 + \Delta} - T_{\mathcal{A}_0}$$

can change scalars.

We can view both bundles as subbundles of $TX \otimes T^*X \otimes T^*X$ of rank $\frac{1}{2} n^2 (n-1)$

plug into second entry of metric ??

Note $\{ \mathfrak{o}(E)\text{-valued 1-forms} \} = \{ \text{section } \Delta^a{}_{bc} \text{ of } TX \otimes T^*X \otimes T^*X :$
$g_{ad} \Delta^d{}_{bc} + g_{db} \Delta^d{}_{ac} = 0 \text{ i.e. } \Delta_{abc} = -\Delta_{bac} \}$

$$\{ E\text{-valued 2-forms} \} = \{ \Delta^a{}_{bc} : \Delta^a{}_{bc} = -\Delta^a{}_{cb} \}$$

And the map $\Delta \mapsto T_{\mathcal{A}_0 + \Delta} - T_{\mathcal{A}_0}$

is $\Delta \longmapsto (\Delta \wedge \Theta)^a_{bc} = \Delta^a{}_{cb} - \Delta^a{}_{bc}$

which is fibrewise linear, so it suffices to prove its a fibrewise isomorphism. Since both have the same rank, its sufficient to prove the map is fibrewise injective

So suppose Δ satisfies $\Delta_{abc} = -\Delta_{bac}$ and it's in the kernel, i.e. $\Delta^{a}{}_{cb} = \Delta^{a}{}_{bc}$. We want to show $\Delta = 0$.

We have $\Delta_{abc} = -\Delta_{bac} = -\Delta_{bca} = \Delta_{cba} = \Delta_{cab} = -\Delta_{acb} = -\Delta_{abc}$

alternately apply 2 equations to cycle indices around.

So $\Delta_{abc} = -\Delta_{abc} \Leftrightarrow \Delta = -\Delta \Rightarrow \Delta = 0$.

$\square$

Given local coordinates on X, get a trivialisation of $E = TX$. The components of the associated local connection 1-forms are the Christoffel symbols. $\Gamma^{i}{}_{jk}$

Definition 8.9: The curvature of the Levi-Civita connection is the Riemann Tensor $R = R^{a}{}_{bcd}$
This is an $\mathfrak{o}(E)$-valued 2-form on X, so we can view it as a tensor of type $(1,3)$.

8.4 Hodge Theory

Let (X, g) be an oriented Riemannian manifold. The metric g induces inner products on each $\Lambda^{p} T^{*}X$.
(if $\alpha_1, \dots, \alpha_n$ are orthonormal 1-forms, then α^{I} give a fibrewise orthonormal basis for $\Lambda^{p} T^{*}X$).

We get a distinguished volume form ω, defined by being positively oriented and of unit length.

Given a p-form β, theres a unique $(n-p)$-form $*\beta$, s.t $\forall$ p-forms α,

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \omega$$

Definition 8.10: The map $* : \Omega^{p}(X) \to \Omega^{p}(X)$ is the Hodge star operator

It's a fibrewise linear isometry $\Lambda^{p} T^{*}X \to \Lambda^{n-p} T^{*}X$ that squares to $(-1)^{p(n-p)}$

Example 8.11: Take $\mathbb{R}^3$ with the standard orientation and metric. So $\omega = dx^1 \wedge dx^2 \wedge dx^3$, and $*dx^1 = dx^2 \wedge dx^3$, and $*dx^1 \wedge dx^2 = dx^3$

Now assume X is compact. Then we can define an inner product on $\Omega^{p}(X)$ via

$$\langle \alpha, \beta \rangle_X = \int_X \langle \alpha, \beta \rangle \omega = \int_X \alpha \wedge *\beta$$

Given a $(p-1)$-form α, p form β, we have

$$\langle d\alpha, \beta \rangle_X = \int_X (d\alpha) \wedge *\beta$$

$$= \int_X \Big(\underbrace{d(\alpha \wedge *\beta)}_{=0 \text{ by Stokes}} - (-1)^{p-1} \alpha \wedge d(*\beta) \Big) \qquad \text{by Leibniz}$$

$$= (-1)^{p} \int_X \alpha \wedge d(*\beta)$$

$$= \langle \alpha, (-1)^{p} *^{-1} d * \beta \rangle_X$$

So the operator $\delta := (-1)^p *^{-1} d * : \Omega^p(X) \to \Omega^{p-1}(X)$ is adjoint to d

Definition 8.12: δ is called the codifferential.
if $\delta\beta = 0$, then β is coclosed
if $\beta = \delta\alpha$, then β is coexact.

(can check $\delta^2 = 0$) easy

Definition 8.13: The Laplace - Beltrami operator is $\Delta := d\delta + \delta d = (d+\delta)^2$
$\Delta : \Omega^p(X) \to \Omega^p(X)$.

If $\Delta\alpha = 0$, then we say α is harmonic.

Write $\mathcal{H}^p$ for the space of harmonic p-forms

Example sheet 4: α is harmonic $\Leftrightarrow$ α is closed and coclosed.

Theorem 8.14 (Hodge): The map $\begin{aligned} \mathcal{H}^p(X) &\longrightarrow H^p_{dR}(X) \\ \alpha &\longmapsto [\alpha] \end{aligned}$ is an isomorphism.

Idea: "$H_{dR}{}^p(X) = {}^{\ker d}/_{\operatorname{Im} d}$
$= \ker d \cap \operatorname{im} d^\perp$
$= \ker d \cap \ker \delta$
$= \mathcal{H}^p(X)$"

Theorem 8.15 (Hodge decomposition)

For all p, $\mathcal{H}^p(X)$ is finite dimensional, and we get orthogonal decompositions

$$\begin{aligned} \Omega^p(X) &= \mathcal{H}^p(X) \oplus \Delta\Omega^p(X) \\ &= \mathcal{H}^p(X) \oplus d\delta\Omega^p(X) \oplus \delta d\Omega^p(x) \\ &= \mathcal{H}^p(X) \oplus d\Omega^{p-1}(x) \oplus \delta\Omega^{p+1}(X) \end{aligned}$$